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ON THE SIEGEL–EISENSTEIN MEASURE AND ITS APPLICATIONS

BY

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ABSTRACT

An Eisenstein measure on the symplectic group over rational number field is constructed which interpolates p-adically the Fourier expansion of Siegel–Eisenstein series. The proof is based on explicit computation of the Fourier expansions by Siegel, Shimura and Feit. As an application of this result a p-adic family of Siegel modular forms is given which interpolates Klingen–Eisenstein series of degree two using Boecherer's integral representation for the Klingen–Eisenstein series in terms of the Siegel–Eisenstein series.

Introduction

Let G be a reductive group over a number field F, and p be a prime number. The arithmetic of L-functions attached to automorphic forms on G, in particular the study of their special values, is closely related to the theory of Eisenstein series via Rankin's method [Ran1, Ran2]. This method uses Eisenstein series in an integral representation for certain rather general complex automorphic L-functions [PSh-R]. In order to construct p-adic automorphic L-functions out of their complex special values one can successfully use p-adic integration along the (two-variable) Eisenstein measure which was introduced by N. Katz [Ka1, Ka2, Ka3] and used by H. Hida [Hi1, Hi2, Hi3] in the case of $G = \mathrm{GL}_2$ over a totally real field F (i.e. for the elliptic modular forms and Hilbert modular forms, see also [Pa1], [Pa3]). The application of such a measure to a given p-adic family of modular forms provides a general construction of p-adic L-functions of several variables. On the other hand, the evaluation of this measure at certain points gives another important source of p-adic L-functions [Ka3].

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The purpose of this paper is to construct a (many-variable) p-adic measure coming from the Siegel–Eisenstein series on the symplectic group

$$G = \mathrm{GSp}_m = \{\alpha \in \mathrm{GL}_{2m} \mid {}^t\alpha J_m \alpha = \nu(\alpha) J_m, \nu(\alpha) \in \mathrm{GL}_1\}$$

over $\mathbf{Q}$ where

$$J_m = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

We use p-adic interpolation of the q-expansions of these series; they were studied by Shimura [ShDu] and Feit [Fe] and have the form

$$E(z; k, \chi, N) = \sum_{\alpha \in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\alpha)) \det(cz+d)^{-k},$$

where z is a variable in the Siegel upper half plane of degree m,

$$\mathfrak{H}_m = \{z \in \mathrm{M}_m(\mathbf{C}) | {}^t z = z = x + iy, \quad y > 0\},$$

$$\Gamma = \Gamma_0^m(N), \quad \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix},$$

and P denotes the subgroup of $P \subset G_{\infty+}$, consisting of elements α with the condition $c_\alpha = 0$, and k is the weight (the above series converges absolutely for $k > m+1$). As a result of our construction we obtain a p-adic measure of $1 + m(m+1)/2$ variables which generalizes the Katz–Eisenstein measure. In the simplest situation the variables (x, ξ) in this family belong to $\mathbf{Z}_p^\times \times A_{m,p}$ where $\mathbf{Z}_p^\times$ is the p-adic unit group, and $A_{m,p}$ is a free $\mathbf{Z}_p$-module of rank $m(m+1)/2$ formed by half integral symmetric matrices of size m over $\mathbf{Z}_p$. The integration of

$$\varphi_{k,\chi} = \det(\xi)^{k-\kappa} x^{k-(m/2)} \chi(x), \quad (x, \xi) \in \mathbf{Z}_p^\times \times A_{m.p})$$

against this measure yields the Siegel–Eisenstein series of weight k with character χ, where $\kappa = (m+1)/2$. Using the variable ξ we can also obtain a twist of the Siegel–Eisenstein series with arbitrary locally constant function of $\varphi(\xi)$ (say with values in $\overline{\mathbf{Q}}$): this twist is a certain (classical) Siegel modular form. The use of an arbitrary p-adic continuous function $\varphi(\xi)$ yields therefore a certain p-adic Siegel modular form (at least when $\varphi(\xi) = \psi(\det)$, where ψ is a Dirichlet character, noted by the referee). In our previous works ([Pa4], [Pa6], [Pa2], [Pa5]) we used a weaker construction of this type in order to build the non-Archimedean standard L-functions of Siegel modular forms for m even and for a sufficiently large weight of a fixed Siegel eigenform. Although these L-functions depend only on one (cyclotomic) variable, the construction worked well both in the p-ordinary and in the "supersingular" cases.

Let $\Lambda = \mathbf{Z}_p[[X]]$ be the Iwasawa algebra. Our construction defines a family of Λ-adic modular forms whose special values are the Siegel–Eisenstein series. A general theory of Λ-adic forms was developed by H. Hida in the elliptic modular case; he then extended it to $G = \mathrm{GL}_n$. Recently a serious attempt to extend this theory to the Siegel modular case was made by K. Buecker (Dissertation of Cambridge University, UK, 1994, under the direction of Prof. R. Taylor), and by J. Tilouine and E. Urban [Ti–U]. If the techniques of Λ-adic forms work for $G = \mathrm{GSp}_m$ as well as for GL_2, one can expect to obtain a much more general construction of non-Archimedean standard L-functions of several variables (at least in the p-ordinary case). Following an idea of A. Wiles, another interesting application of the Siegel–Eisenstein measure is related to an explicit construction of Λ-adic Siegel modular forms using multiplication of a given form f by a family G_k of (not necessarily holomorphic) Eisenstein series, then applying to it the holomorphic projection operator $\mathcal{H}ol$ of Section 2, and decomposing the resulting family into a sum of Λ-eigenforms. On the other hand, for a homogeneous polynomial function $P(\xi) \in \mathbf{Z}[\xi]$ one can attach a differential operator $P(D)$ where D is the matrix whose entries are the derivations $(1/2)(1+\delta_{i,j})\partial/\partial z_{i,j}$, and $z = (z_{i,j})$ is a variable symmetric matrix. A certain twist of $P(D)$ produces a non-holomorphic differential operator Δ_P which sends automorphic forms to automorphic forms and preserves the arithmeticity at CM-points [ShAJ]. Another characteristic feature is that it maps a holomorphic Eisenstein series G to a non-holomorphic Eisenstein seies $\Delta_P G$. In Section 2 we describe a complex analytic family of the type $\mathcal{H}ol\,(f \cdot \Delta_P G)$ which should correspond to a p-adic family coming from the product of $P(\xi)$ by the Siegel–Eisenstein measure. We intend to describe in detail this p-adic family in another paper. It would be interesting to describe explicitly the algebra of p-adic differential operators acting on the spaces of p-adic Siegel modular forms and to understand its interrelation with the p-adic Hecke algebra keeping in mind that in the elliptic modular case for $m = 1$ the Ramanujan operator commutes with the action of the Hecke operators of the appropriate weight.

It was pointed out by the referee that it would be interesting to study a possibility of evaluation of this Siegel–Eisenstein measure at CM points. This could be done only when the measure has values in the space of p-adic modular forms not just in the monoid ring R used in this paper. However, some additional analysis on this point would be necessary and could be the subject of another paper.

CONTENT OF THE PAPER. In the first two sections we discuss rationality properties of Fourier coefficients of the Siegel–Eisenstein series; here we give some known techniques of computing their Fourier expansions and the action of the holomorphic projection operator and the Maass differential operator on Fourier expansions of Siegel modular forms. In Section 3 we recall some basic facts from the theory of non-Archimedean integration including the S-adic Mazur measure and its Mellin transform. Most of this material can be found in [Pa4]. Main theorems (Theorem 4.3 and 4.4) in the cases of even and odd degree m respectively are formulated and proven in Section 4. In Section 5, written in cooperation with Koji Kitagawa (Hokkaido University, Japan), we describe an application to the Λ-adic Klingen–Eisenstein series and we construct a p-adic measure coming from the Klingen–Eisenstein series on the symplectic group.

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1. Formulas for Fourier coefficients of the Siegel–Eisenstein series

1.1. RATIONALITY PROPERTIES OF FOURIER COEFFICIENTS OF THE SIEGEL–EISENSTEIN SERIES. We start by recalling the definition of these series. We call matrices $c, d \in \mathrm{M}_m(\mathbf{Z})$ coprime iff

$$\{G \in \mathrm{M}_m(\mathbf{Q}) | Gc,\ Gd \in \mathrm{M}_m(\mathbf{Z})\} = \mathrm{M}_m(\mathbf{Z}).$$

A couple (c, d) is called **a symmetric couple** if $c^t d = d^t c$. Two symmetric couples of coprime matrices are called **equivalent** iff for some unimodular matrix $U \in \mathrm{GL}_m(\mathbf{Z})$ we have $(c_1, d_1) = (Uc_2, Ud_2)$.

Let $\Delta = \Delta_m$ denote the set of equivalence classes of symmetric couples of coprime matrices. Then the set can be identified with the set of right coset classes $\Gamma_0^m \backslash \Gamma^m$ of the group $\Gamma^m = \mathrm{Sp}_m(\mathbf{Z})$ with respect to its parabolic subgroup

$$\Gamma_0^m = \left\{ \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} | \gamma \in \Gamma^m \right\}$$

via the map

$$\Gamma_0^m \backslash \Gamma^m \ni \Gamma_0^m \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \text{class of } (c, d) \in \Delta_m. \tag{1.1}$$

We see also that via this map the set

$$\{(c,d) \in \Delta_m | c \equiv 0(\mathrm{mod} N)\}$$

is identified with a coset for $\Gamma_0^m \backslash \Gamma_0^m(N)$.

Now let k, N be positive integers, s a complex number and χ a Dirichlet character modulo N such that $\chi(-1) = (-1)^k$. For $z \in \mathfrak{H}$ the Siegel–Eisenstein series is defined by

$$E(z,s;k,\chi,N) = E(z,s) = \det(y)^s \sum \chi(\det(d)) \det(cz+d)^{-k-|2s|}, \tag{1.2}$$

where the summation is taken over all $(c,d) \in \Delta$ with the condition $c \equiv 0(\mathrm{mod}\ N)$ and we use the convenient notation by Deligne and Ribet [De-R]:

$$z^{-k-|2s|} \stackrel{\mathrm{def}}{=} z^{-k}|z|^{-2s} \quad \text{for } z \in \mathbf{C}^*.$$

The series (1.2) is absolutely convergent for $k + 2\,\mathrm{Re}(s) > m+1$ and it admits a meromorphic analytic continuation over the whole complex s-plane. Put

$$j(\alpha, z) = \det(cz+d) \quad \text{for } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } z \in \mathfrak{H},$$

then it follows from the description of Δ_m given above that

$$E(z,s;k,\chi,N) = \det(y)^s \sum_{\alpha \in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\alpha)) j(\alpha,z)^{-k-|2s|}, \tag{1.3}$$

where

$$\Gamma = \Gamma_0^m(N), \quad \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix},$$

and P denotes the subgroup of $P \subset G_{\infty+}$, consisting of elements α with the condition $c_\alpha = 0$.

For the full symplectic modular group $\Gamma = \mathrm{Sp}_m(\mathbf{Z})$ these series were defined by Siegel [Sie].

In the original definition by Siegel the number k is even and $k > m+1$, so that the series (1.2) is absolutely convergent and is referred to as the Siegel–Eisenstein series. The rationality property of its Fourier coefficients was established by Siegel himself (although it was certainly known earlier in the case $m = 1$:

$$\begin{aligned} E_k^{(1)}(z) &= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e(nz) \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{2\sigma_{k-1}(n)}{\zeta(1-k)} \right) e(nz), \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}, \end{aligned} \tag{1.4}$$

where B_k are Bernoulli numbers, $\zeta(s)$ being the Riemann zeta function). After Siegel's original work his calculation was generalized in various directions: to the case of congruence subgroups of $\Gamma_0^{(m)}(N) \subset \Gamma^m$ [St], to non-convergent series defined by analytic continuation over an additional parameter (Hecke's method) [Fe], to other classes of algebraic groups and symmetric domains [Bö2], [Har1], [Fe], [ShDu], [Sh82]. It was discovered that the rationality property remains valid even for more general series of Eisenstein type (the **Klingen–Eisenstein series**).

In this paper we are interested in the Siegel–Eisenstein series defined by (1.2) for $k + 2\,\mathrm{Re}\,(s) > m+1$ (we use again the notation $z^{-k-|2s|} = z^{-k}|z|^{-2s}$ for $z, s \in \mathbf{C}, k \in \mathbf{Z}$) and by analytic continuation over s for other values of $s \in \mathbf{C}$. It is assumed in the identity (1.2) that $N > 1, \chi$ is a Dirichlet character mod N (not necessarily primitive, e.g. trivial modulo $N > 1$), and

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in \Gamma = \Gamma_0^m(N) \subset \Gamma^m.$$

In the following study of certain arithmetic properties of Fourier coefficients we use an explicit calculation of the Fourier expansion of the series

$$E^*(z,s) = E(-z^{-1}, s)(\det z)^{-k}, \tag{1.5}$$

obtained from (1.2) by applying the involution

$$\eta = J_m = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix}.$$

However, for $k > m+1$ and $N = 1$ both series coincide and reduce to the series originally studied by Siegel:

$$E(z) = E_k^m(z) = E(z,0) = E^*(z,0).$$

The detailed study of these series was conducted by Shimura [ShDu] and P. Feit ([Fe], §10) in a more general situation, in particular, for the case of Eisenstein series attached to the group Sp_m over a totally real field. For convenience we reproduce only a specialization of these results to the case of $F = \mathbf{Q}$.

1.2. Preparation: the confluent hypergeometric function. For a detailed description of the Fourier expansion of the series (1.5) we need some additional notation. Let

$$V = V_m = \{h \in \mathrm{M}_m(\mathbf{R}) | {}^t h = h\} \tag{1.6}$$

be the set of all real symmetric matrices of size $m \times m$, and

$$Y = V^+ = \{h \in V | h > 0\} \tag{1.7}$$

the subset of its positive definite elements. For each matrix $T \in \mathrm{M}_m(\mathbf{R})$ let $\delta_+(T)$ denote the product of all positive eigenvalues of T, $\delta_-(T) = \delta_+(-T)$ and $\delta_+(T) = 1$ if T does not have positive eigenvalues.

For $h \in V$ let $p = p(h)$ denote the number of positive eigenvalues of h counted with their multiplicities, and $q = q(h)$ the number of negative eigenvalues. Then $r = r(h) = p + q$ is the rank of h.

Let also

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2)) \tag{1.8}$$

be the Γ-function of degree m. This function generalizes the usual Γ-function in view of the following integral representation:

$$\Gamma_m(s) = \int_Y (\det y)^s e^{-\operatorname{tr}(y)} d^\times y, \tag{1.9}$$

which is valid for $s \in \mathbf{C}$ with $\mathrm{Re}\,(s) > (m-1)/2$, and

$$dy = \prod_{i \le j} dy_{ij}, \quad d^\times y = \det(y)^{-(m+1)/2} dy.$$

Recall that $d^\times y$ is a measure on Y which is invariant with respect to the action of $a \in \mathrm{GL}_m(\mathbf{R})$ given by $d^\times({}^t aya) = d^\times y$. For complex numbers α and β we define the numbers

$$\begin{aligned} \kappa = (m+1)/2, \quad \tau &= \tau(h, \alpha, \beta) \\ &= (2p - m)\alpha + (2q - m)\beta + m + (m - r)\kappa + pq/2, \end{aligned} \tag{1.10}$$

$$\sigma = \sigma(h, \alpha, \beta) = p\alpha + q\beta + m - r + \{(m - r)(m - r - 1) - pq\}/2. \tag{1.11}$$

In [Sh82] Shimura studied the confluent hypergeometric function

$$\omega(y, h; \alpha, \beta), \tag{1.12}$$

which is defined for all $(y, h; \alpha, \beta) \in Y \times V \times \mathbf{C}^2$ and which is holomorphic in variables $(\alpha, \beta) \in \mathbf{C}^2$. It can be used for computing the Fourier expansion of the series

$$S(z, L; \alpha, \beta) = \sum_{a \in L} \det(z + a)^{-\alpha} \det(\overline{z} + a)^{-\beta} \quad (z \in \mathfrak{H}_m), \tag{1.13}$$

which is obtained by summation over a lattice $L \subset V$ and is absolutely convergent for $\operatorname{Re}(\alpha + \beta) > m$. Let

$$L' = \{h \in V \,|\, \operatorname{tr}(hL) \in \mathbf{Z}\}$$

be the lattice dual to L with respect to the pairing given by

$$(u, v) \mapsto e_m(uv) = \exp(2\pi i \operatorname{tr}(uv)).$$

In particular there is the equality

$$\mu(V/L) S(z; \alpha, \beta) = \sum_{h \in L'} \xi(y, h; \alpha, \beta) e_m(hx) \tag{1.14}$$

in which

$$\mu(V/L) = \int_{V/L} dy$$

denotes the volume of a fundamental domain V/L,

(1.15)
$$\begin{aligned} \xi(y, h; \alpha, \beta) = & i^{m\beta - m\alpha} 2^{\tau} \pi^{\sigma} \Gamma_{m-r}(\alpha + \beta - \kappa) \Gamma_{m-q}(\alpha)^{-1} \Gamma_{m-p}(\beta)^{-1} \\ & \times (\det y)^{\kappa - \alpha - \beta} \delta_+(hy)^{\alpha - \kappa + q/4} \delta_-(hy)^{\beta - \kappa + q/4} \omega(2\pi y, h; \alpha, \beta), \end{aligned}$$

and it is additionally assumed that $\operatorname{Re}(\alpha) > m/2$, $\operatorname{Re}(\beta) > m/2$ (for the regularity of Γ-functions in (1.15) see [Sh82], (4.34.K)) and we adopt the standard choice of branch for the exponentiation, namely,

$$v^{\alpha} = e^{\alpha \log(v)}, \quad -\pi \leq \operatorname{Im}(\log v) < \pi.$$

The function $\xi(y, h; \alpha, \beta)$ admits the following integral representation: for $g \in Y$, $h \in V, (\alpha, \beta) \in \mathbf{C}^2$

$$\xi(y, h; \alpha, \beta) = \int_V e_m(-hx) \det(x + iy)^{-\alpha} \det(x - iy)^{-\beta} dx, \tag{1.16}$$

with the integral being absolutely convergent for $\operatorname{Re}(\alpha + \beta) > 2\kappa - 1$ (see [Sh82], (1.25)).

Applying the equality (1.14) to the lattice $L = S = V \cap \mathrm{M}_m(\mathbf{Z})$ when $L' = A = A_m$ is the lattice of all symmetric half integral matrices and also to the lattice $L = NS, L' = N^{-1}A_m$ with $\alpha = k$, $\beta = 0, k > m$ and $C_m = A_m \cap Y$, we get the classical equality

$$\sum_{a \in S} \det (z+a)^{-k} = (2\pi i)^{mk}\Gamma_m(k)^{-1} \sum_{h \in C_m} (\det h)^{k-\kappa} e_m(hz). \tag{1.17}$$

Indeed, we notice that the only terms in equality (1.17) correspond to $p = m$, $q = 0$ because of the poles of the Γ-functions in the denominator of (1.15). Also, the function $\omega(2\pi y, h; \alpha, \beta)$ reduces to the exponent $e_m(iyh)$ in view of the formulas

$$\xi(y, h; \alpha, 0) = i^{-m\alpha}2^{(1-\kappa)m}(2\pi)^{m\alpha}\Gamma_m(\alpha)^{-1}(\det h)^{\alpha-\kappa}e_m(iyh), \tag{1.18}$$

$$\xi(y, 0; \alpha, \beta) = i^{m\beta - m\alpha}2^{m(\kappa+1-\alpha-\beta)}\pi^{m\kappa}\frac{\Gamma_m(\alpha+\beta-\kappa)}{\Gamma_m(\alpha)\Gamma_m(\beta)}, \tag{1.19}$$

$$\lim_{s\to 0} \xi(y, h; \kappa + s, s) = i^{-m\kappa}2^{\theta}\pi^{m\kappa}\Gamma_m(\kappa)^{-1}e_m(iyh), \tag{1.20}$$

with $q = 0$ and $\theta = [(m+p)/2]$ (see also [ShDu], (7.11)–(7.14)).

The confluent hypergeometric function $\omega(2\pi y, h; \alpha, \beta)$ can be used for analytic continuation of the Siegel–Eisenstein series [Fe], [ShDu] by means of the term-by-term analytic continuation of their Fourier coefficients. These coefficients can be expressed in terms of the functions (1.12) (see Theorem 1.6 below). We list also some other properties of these functions, which are useful for the analytic continuation (see [Sh82], theorem 4.2):

functional equation

$$\omega(2\pi y, h; \alpha, \beta) = \omega(2\pi y, h; \kappa + (t/2) - \beta, \kappa + (t/2) - \alpha), \tag{1.21}$$

where $t = m - r$;

a uniform upper bound on compact subsets

$$|\omega(2\pi y, h; \alpha, \beta)| \le C_1 e^{-\tau(hy)}(1 + \mu(hy)^{-C_2}), \tag{1.22}$$

with α, β varying in a fixed compact subset $T \subset \mathbf{C}^2$ and the constants C_1, C_2 depending only on $T, \tau(x)$ being *the sum of eigenvalues* of a matrix x, μ *the minimum of their absolute values.*

1.3. Critical values of the confluent hypergeometric function. Now we give formulas, which express the function $\omega(2\pi y, h; \alpha, \beta)$ in terms of certain polynomials of the entries of the matrix $y = (y_{ij})$ provided $h > 0$ and either $\alpha - \kappa \in \mathbf{Z}, \alpha - \kappa \geq 0$ or $\beta \in \mathbf{Z}, \beta \leq 0$ ($\kappa = (m+1)/2$). The pairs (α, β) satisfying these conditions will be called **critical**: as we shall see, the critical values of s for the standard zeta function correspond to certain critical pairs. The following calculation of the special values is based on properties of the function $\zeta(z; \alpha, \beta)$ defined for $z \in \mathfrak{H}' = \{z \in \mathrm{M}_m(\mathbf{C}) | iz \in \mathfrak{H}\}$ by the integral

$$\zeta(z; \alpha, \beta) = \int_Y e^{-\operatorname{tr}(zx)} \det(x + 1_m)^{\alpha-\kappa} \det x^{\beta-\kappa} dx, \tag{1.23}$$

which is absolutely convergent for Re $> \kappa - 1$ and defines a holomorphic function of (z, α, β). Let

$$\omega(z; \alpha, \beta) = \Gamma_m(\beta)^{-1} \det(z)^{\beta} \zeta(z; \alpha, \beta). \tag{1.24}$$

It was established by Shimura ([Sh82], theorem 3.1) that the function (1.24) can be analytically continued to a holomorphic function over $\mathfrak{H}' \times \mathbf{C}^2$ satisfying the functional equation

$$\omega(z; \kappa - \beta, \kappa - \alpha) = \omega(z; \alpha, \beta). \tag{1.25}$$

For an arbitrary compact subset $T \subset \mathbf{C}^2$ there exist positive constants $A, B > 0$ depending only on T such that

$$|\omega(z; \alpha, \beta)| \leq A(1 + \mu(y)^{-B}) \quad \text{for } y \in Y \subset \mathfrak{H}', \quad (\alpha, \beta) \subset T. \tag{1.26}$$

It is known also (see [Sh82], (4.19)) that

$$\omega(y, 1_m; \alpha, \beta) = 2^{-m(m+1)/2} e^{-\operatorname{tr}(y)} \omega(2y; \alpha, \beta) \tag{1.27}$$

and that for all $a \in \mathrm{SL}_m(\mathbf{R})$ one has

$$\omega({}^t a^{-1} y a^{-1}; \alpha, \beta) = \omega(y; \alpha, \beta), \tag{1.28}$$

$$\omega(y, -h; \alpha, \beta) = \omega(y, h; \beta, \alpha), \tag{1.29}$$

$$\omega(y, h; \alpha, \beta) = 1. \tag{1.30}$$

Comparison of (1.27) and (1.28) shows that for $h > 0$ there is the identity

$$\begin{aligned}\omega(y,h;\alpha,\beta) &= \omega(a(hy)a^{-1},1_m;\alpha,\beta) \\ &= 2^{-m(m+1)/2}e^{-\operatorname{tr}(y)}\omega(2a(h^{1/2}yh^{1/2})a^{-1};\alpha,\beta).\end{aligned} \tag{1.31}$$

Now let us consider the differential operator Δ_m (the **Maass differential operator**) acting on C^∞-functions over $V\otimes\mathbf{C}$ of degree m, which is defined by the equality

$$\Delta_m = \det(\partial_{ij}), \quad \partial_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial z_{ij}. \tag{1.32}$$

For an integer $n \geq 0$ and a complex number β consider the polynomial

$$R(z;n,\beta) = (-1)^{mn}e^{\operatorname{tr}(z)}\det(z)^{n+\beta}\Delta_m^n[e^{-\operatorname{tr}(z)}\det(z)^{-\beta}], \tag{1.33}$$

with $z \in V\otimes\mathbf{C}$, where the exponentiation is well-defined by

$$\det(y)^\beta = \exp(\log(\det(y))) \quad \text{for } \det(y)>0, \quad y\in Y\otimes\mathbf{C}.$$

According to definition (1.33) the degree of the polynomial $R(z;n,\beta)$ is equal to mn and the term of the highest degree coincides with $\det z^n$. We have also that for $\beta\in\mathbf{Q}$ the polynomial $R(z;n,\beta)$ has rational coefficients.

PROPOSITION (See [Sh82], proposition 3.2): *For any non-negative integer n the functions $\det(z)^n\omega(z;n+\kappa,\beta)$ and $\det(z)^n\omega(z;\alpha,-n)$ are polynomial functions of z. More precisely, we have that*

$$\omega(z;n+\kappa,\beta) = \det(z)^{-n}R(z;n,\beta), \tag{1.34}$$

$$\omega(z;\alpha,-n) = \omega(z;n+\kappa,n-\alpha) = \det(z)^{-n}R(z;n,\kappa-\alpha). \tag{1.35}$$

Notice also that for $m=1$ one has

$$R_1(z,n,\beta) = \sum_{k=0}^{n}\binom{n}{k}\beta(\beta+1)\cdots(\beta+k-1)z^{n-k}. \tag{1.36}$$

In general $R_m(z;n,\beta)$ can be expressed in terms of a polynomial with rational coefficients of $\lambda_r(z)$, the polynomial functions of entries of the matrix variable $z\in\mathrm{M}_m(\mathbf{C})$ defined by

$$\det(t1_m - X) = \sum_{r=0}^{m}(-1)^r\lambda_r(X)t^{m-r}. \tag{1.37}$$

If we apply this to functions $\omega(2\pi y, h; \alpha, \beta)$ from (1.21) then we get for $h > 0$ the following identity:

$$
\begin{aligned}
\omega(2\pi y, h; n+\kappa, \beta) &= \omega(2\pi y, h; \kappa-\beta, n) \\
&= 2^{-m(m+1)/2} e_m(ihy)\omega(4\pi a^{-1}(hy)a\,; n+\kappa, \beta) \\
&= 2^{-m(m+1)/2} e_m(ihy) \det(4\pi hy)^{-n} R_m(4\pi hy\,; n, \beta).
\end{aligned} \tag{1.38}
$$

1.4. PROPOSITION (Fourier expansion of the Siegel–Eisenstein series $E^*(z,s)$, see [Fe], §10): *The series defined by (1.5) have the following Fourier expansion:*

$$
E^*(z,s) = \sum_{h \in N^{-1}A_m} b(h,y,s) e_m(hz), \tag{1.39}
$$

in which coefficients have the form of the product

$$
b(h,y,s) = N^{-m\kappa} W(y,h,s)\Gamma(h,s) RL^*(h,\chi,k+2s) M(h,\chi,k+2s), \tag{1.40}
$$

with the factors described as follows ((a)–(d))*:*
(a) *The confluent hypergeometric function*

$$
\begin{aligned}
W(y,h,s) =& i^{-mk} 2^{\tau}\pi^{\sigma}\omega(2\pi y, h; k+s, s) \\
& \times (\det y)^{\kappa-k-s}\delta_+(hy)^{k+s-\kappa+q/4}\delta_-(hy)^{s-\kappa+p/4},
\end{aligned} \tag{1.41}
$$

with (compare with (1.15))

$$
\begin{aligned}
\tau &= (2p-m)(k+s) + (2q-m)s + m + (m-r) + pq/2 \\
&= 2(r-m)s + (2p-m)k + m + (m-r)(m+1)/2 + pq/2, \\
\sigma &= p(k+s) + qs + m - r + \{(m-r)(m-r-1) - pq\}/2 \\
&= rs + pk + \{(m-r)(m-r-1) - pq\}/2.
\end{aligned}
$$

(b) *Gamma factor $\Gamma(h,s)$. Let, for integer r, the symbol $\varepsilon(r)$ denote its parity: $\varepsilon(r) = 0$, 1 with $r \equiv \varepsilon(r)$* mod 2. *Put $\delta = \varepsilon(k), \mu = \varepsilon((r/2)+q+k)$ and then define:*

for $\varepsilon(r) = 0$,

$$
\Gamma(h,s) = \frac{\Gamma_{m-r}\left(k+2s-\frac{m+1}{2}\right)\Gamma\left(s+\frac{k+\delta}{2}\right)\prod_{i=1}^{[m/2]}\Gamma(k+2s-i)}{\Gamma_{m-q}(k+s)\Gamma_{m-p}(s)\Gamma\left(s+\frac{k-m+r/2+\mu}{2}\right)\prod_{i=1}^{[(m-r)/2]}\Gamma(k+2s-m+i+(r-1)/2)};
$$

for $\varepsilon(r) = 1$,

$$\Gamma(h,s) = \frac{\Gamma_{m-r}\left(k+2s-\frac{m+1}{2}\right)\Gamma\left(s+\frac{k+\delta}{2}\right)\prod_{i=1}^{[m/2]}\Gamma(k+2s-i)}{\Gamma_{m-q}(k+s)\Gamma_{m-p}(s)\prod_{i=1}^{[(m-r-1)/2]}\Gamma(k+2s-m+i+r/2)}.$$

(c) *The ratio of Dirichlet L-functions* RL^*. *Let, for a Dirichlet character* χ *modulo* N *of parity* $\delta = 0$ *or 1,*

$$L_N^*(s,\chi) = \Gamma((k+\delta)/2)L_N(s,\chi) = \Gamma((k+\delta)/2)\prod_{q \nmid N}(1-\chi(q)q^{-s})^{-1} \tag{1.42}$$

denote the normalized Dirichlet L-function, which is regular for all $s \in \mathbf{C}, s \neq 1$, *including* $s = 0$ *(due to the condition* $N > 1$*). Next we define an additional quadratic Dirichlet character* χ_h *depending on* $h \in A_m$ *and defined only for even* $r \neq 0$. *Namely, for* $h = 0$ *let* $\chi_h = \chi_0$ *be trivial; for* $h \neq 0$ *we know that for some matrix* $u \in \mathrm{GL}_m(\mathbf{Q})$,

$$^t uhu = \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } \det\ h_1 \neq 0; \tag{1.43}$$

then let χ_h *denote the quadratic character attached to the quadratic field* $\mathbf{Q}(\sqrt{\det h_1})/\mathbf{Q}$ *(this definition does not depend on the choice of a matrix* u*). Put*

$$\theta(a) = \left(\frac{-1}{a}\right). \tag{1.43a}$$

Under this notation we set:

for an even r *(i.e. with* $\varepsilon(r) = 0$*),*

$$RL^*(h,\chi,k+2s) = \frac{L_N^*(k+2s-m+(r/2),\chi\theta^{r/2}\chi_h)\prod_{i=1}^{[(m-r)/2]}L_N^*(2k+4s-2m+r-1+2i,\chi^2)}{L_N^*(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N^*(2k+4s-2i,\chi^2)};$$

for an odd r *(i.e. with* $\varepsilon(r) = 1$*),*

$$RL^*(h,\chi,k+2s) = \frac{\prod_{i=1}^{[(m-r-1)/2]}L_N^*(2k+4s-2m+r-1+2i,\chi^2)}{L_N^*(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N^*(2k+4s-2i,\chi^2)}.$$

(d) *The integral factor*

$$M(h,\chi,k+2s)=\prod_{q\in P(h)} M_q(h,\chi(q)q^{-k-2s}) \tag{1.44}$$

is a finite Euler product, extended over primes q in the set $P(h)$ of prime divisors of the number N and of all elementary divisors of the matrix h. The important property of the product is that for each q we have that $M_q(h,t)\in \mathbf{Z}[t]$ is a polynomial with integral coefficients.

The explicit form of this polynomial is insignificant for our purposes; however, one can find interesting explicit formulas for these coefficients in [Rag], [KiY].

1.5. Normalized Siegel–Eisenstein series. We introduce here three types of normalized Siegel–Eisenstein series in order to give a precise statement on their holomorphy properties with respect to the variable s, the properties of positivity of matrices ξ by which their Fourier coefficients are indexed, and also algebraic properties of these Fourier coefficients:

$$\begin{aligned}G^*(z,s)&=G^*(z,s;k,\chi,N)\\&=N^{m(k+2s)/2}i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma(1_m,s)^{-1}\\&\quad\times L_N^*(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N^*(2k+4s-2i,\chi^2)E(-(Nz)^{-1},s)\det(\sqrt{N}z)^{-k}\\&=N^{m(k+2s)/2}\tilde\Gamma(k,s)L_N(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N(2k+4s-2i,\chi^2)E^*(Nz,s),\end{aligned}\tag{1.45}$$

with

$$\begin{aligned}E^*(Nz,s)&=E(-(Nz)^{-1},s)\det(Nz)^{-k}=N^{-km/2}E|W(N),\\\tilde\Gamma(k,s)&=i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma(1_m,s)^{-1}\\&\qquad\times\Gamma((k+2s+\delta)/2)\prod_{j=1}^{[m/2]}\Gamma(k+2s-j)\\&=i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\times\begin{cases}\Gamma_m(k+s)\,\Gamma(s+(k-\frac{m}{2}+\mu)/2), & \text{if } m \text{ is even;}\\ \Gamma_m(k+s), & \text{otherwise.}\end{cases}\end{aligned}\tag{1.46}$$

If m is odd then we put $G^+(z,s)=G^-(z,s)=G^*(z,s)$. If m is even then we define (with $\mu=\varepsilon((m/2)+k)$)

$$G^-(z,s)=\Gamma((k+2s-(m/2)+\mu)/2)^{-1}G^*(z,s),\tag{1.47}$$

(1.48)
$$\begin{aligned}G^+(z,s) =&\frac{i^{\mu}\pi^{(1/2)-k-2s+(m/2)}}{\Gamma((1-k-2s+(m/2)+\mu)/2)}G^*(z,s)\\ =&\frac{2i^{\mu}\Gamma(k+2s-(m/2))\cos(\pi(k+2s-(m/2)-\mu)/2)}{(2\pi)^{k+2s-(m/2)}}G^-(z,s).\end{aligned}$$

Note that the normalizing factors in formulas (1.45), (1.47) and (1.48) are closely related to those of the Dirichlet L-series and the standard zeta functions (for even m).

For the normalized series $G^*(z,s)$ we have

$$G^*(z,s) = \sum_{h\in A_m} b^*(h,y,s)e_m(hz), \tag{1.49}$$

where

$$b^*(h,y,s) = W^*(y,h,s)\Gamma^*(h,s)L_N^*(h,\chi,k+2s)M(h,\chi,k+2s),$$

with

$$\begin{aligned}\Gamma^*(h,s) =&\Gamma^-(1_m,s)^{-1}\Gamma^-(h,s),\\ W^*(y,h.s) =&N^{m(s+k-\kappa)}i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}e_m(-ihy)W(Ny,N^{-1}h,s)\\ =&i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}e_m(-ihy)W(y,h,s).\end{aligned}$$

The factor $M(h,\chi,k+2s)$ is given by (1.44), and for r even we have

$$L_N^*(h,\chi,k+2s) =$$
$$L_N^*(k+2s-m+(r/2),\chi\theta^{r/2}\chi_h)\prod_{i=1}^{[(m-r)/2]} L_N^*(2k+4s-2m+r-1+2i,\chi^2)$$

and for r odd

$$L_N^*(h,\chi,k+2s) = \prod_{i=1}^{[(m-r-1)/2]} L_N^*(2k+4s-2m+r-1+2i,\chi^2).$$

1.6. THEOREM (on Fourier coefficients with positive matrix indices):

(a) *Suppose that $N>1$ and let m be even, $2k>m$. Then: If $2s$ is an integer such that $s\le 0, k+2s\ge 1+(m/2)$, there is the following Fourier expansion:*

$$G^+(z,s) = \sum_{A_m\ni h>0} b^+(h,y,s)e_m(hz), \tag{1.50}$$

where for $s > (m+2-2k)/4$ in (1.50) non-zero terms only occur for positive definite $h > 0$, and for all s from (a) with $h > 0$, $h \in A_m$ the following identity holds:

$$b^+(h,y,s) = W^*(y,h,s)L_N^+(k+2s-(m/2), \chi\theta^{m/2}\chi_h)M(h,\chi,k+2s),$$

where

$$L^+(s,\chi) = \frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}L_N(s,\chi)$$

is the normalized Dirichlet L-function, $\delta = 0$ or 1 according to $\chi(-1) = (-1)^\delta$, the factor $M(h,\chi,k+2s)$ is defined by (1.44), χ_h and θ defined by (1.43) and (1.43a),

$$W^*(y,h,s) = 2^{-m\kappa}\det h^{k+2s-\kappa}\det(4\pi y)^s R(4\pi hy; -s; \kappa-k-s),$$

provided s is an integer, where $R(y;n,\beta)$ is defined by (1.33), and $b^+(h,y,s) = 0$ otherwise (if $2s \in \mathbf{Z}$ but $s \notin \mathbf{Z}$).

(b) *Let m be odd, $2k > m$. Then: If $2s$ is an integer such that $s \le 0$, $k+2s \ge 1+(m/2)$, there is the following Fourier expansion:*

$$G^+(z,s) = \sum_{A_m \ni h > 0} b^+(h,y,s)e_m(hz), \tag{1.51}$$

where for $s > (m+2-2k)/4$ in (1.50) non-zero terms only occur for positive definite $h > 0$, and for all s from (a) with $h > 0$, $h \in A_m$ the following identity holds:

$$b^+(h,y,s) = W^*(y,h,s)M(h,\chi,k+2s),$$

where the factor $M(h,\chi,k+2s)$ is defined by (1.44),

$$W^*(y,h,s) = 2^{-m\kappa}\det h^{k+2s-\kappa}\det(4\pi y)^s R(4\pi hy; -s; \kappa-k-s),$$

provided s is an integer, where $R(y;n,\beta)$ is defined by (1.33), and $b^+(h,y,s) = 0$ otherwise (if $2s \in \mathbf{Z}$ but $s \notin \mathbf{Z}$).

The proof is deduced from the expansions (1.49) using the definition of the normalizing factors, see [Pa4], Ch. 2, Theorem 3.8. Note that by (1.41)

$$\begin{aligned} W^*(y,h,s) =& e_m(-ihy)\omega(2\pi y,h;k+s,s)(\det y)^{\kappa-k-s} \\ &\times \delta_+(hy)^{k+s-\kappa+q/4}\delta_-(hy)^{s-\kappa+p/4}, \end{aligned} \tag{1.52}$$

and then take into account formula (1.38) for the critical values of the function ω. In the case of odd parity of $2s \in \mathbf{Z}$ the corresponding Fourier coefficients vanish by properties of the Γ-factors in (1.47), (1.48).

2. Holomorphic projection operator and the Maass differential operator

2.1. Holomorphic projection operator. Recall that a function

$$F\colon \mathfrak{H}_m \to \mathbf{C}, \quad F \in C^\infty(\mathfrak{H}_m)$$

is called a **C^∞-modular form** of weight k on the group $\Gamma_0^m(N)$ with a Dirichlet character ψ mod N if

$$F((az+b)(cz+d)^{-1}) = \psi(\det\ d)\det(cz+d)^k F(z) \quad \forall\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(N).$$

The complex vector space of functions F with the above condition will be denoted by $\tilde{\mathcal{M}}_m^k(N,\psi)$. For all $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ there is the following Fourier expansion:

$$F(z) = \sum_{h\in A_m} A(y,h)e_m(hx), \tag{2.1}$$

where $A(y,h)$ are some C^∞-functions on Y. The Petersson inner product is defined for an arbitrary holomorphic cusp form $f \in \mathcal{S}_m^k(N,\psi)$ and $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ by

$$\langle f,\, F\rangle_N = \int_{\Phi_0(N)} \overline{f(z)}F(z)\ \det\ y^{k-m-1}dx\,dy,$$

where $\Phi_0(N) = \Gamma_0^m(N)\backslash\mathfrak{H}_m$ is a fundamental domain for the group $\Gamma_0^m(N)$.

We call a function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ **a function of bounded growth** if for each $\varepsilon > 0$ the following integral converges:

$$\int_X\int_Y |F(z)|\det\ y^{k-1-m}e^{-\varepsilon\,\mathrm{tr}(y)}dy\,dx < \infty \tag{2.2}$$

where

$$X = \{x \in M_m(\mathbf{R}) | {}^t x = x, |x_{ij}| \le 1/2 \text{ for all } i,j\},$$
$$Y = \{y \in M_m(\mathbf{R}) | {}^t y = y > 0\}.$$

Respectively, we call a function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ **a function of a moderate growth** if for all $z \in \mathfrak{H}$ and for all sufficiently large values of $\mathrm{Re}\,(s) \gg 0$ the integral

$$\int_{\mathfrak{H}} F(w)\det(\overline{w}-z)^{-k-|2s|}\det\mathrm{Im}\,(w)^{k+s}d^\times w \tag{2.3}$$

is absolutely convergent and admits an analytic continuation over s to the point $s = 0$. The last definition may differ from a traditional one; its meaning is clarified by the following result (Theorem 2.2), which provides a refinement of theorem 1 of Sturm's paper [St]. It will follow from the proof that all functions of bounded growth automatically turn out to be of moderate growth in the sense of definitions (2.2), (2.3) given above.

2.2. THEOREM: *Let $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ and $k > 2m$. Put for $h > 0$, $h \in A_m$*

$$a(h) = c(k,m)^{-1} \det(4h)^{k-(m+1)/2} \int_Y A(y,h) e_m(ihy) \det\, y^{k-1-m}\, dy, \tag{2.4}$$

with

$$c(t,m) = \Gamma_m(t-(m+1)/2)\pi^{-m(t-(m+1)/2)},$$

and $A(y,h)$ being coefficients of the expansion (2.1); and suppose that the integral (2.4) is absolutely convergent. Define the function

$$\mathcal{H}ol\, F(z) = \sum_{A_m \ni h > 0} a(h) e_m(hz). \tag{2.5}$$

Then
(a) *if the function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ is of bounded growth then $\mathcal{H}ol\, F(z) \in \mathcal{S}_m^k(N,\psi)$;*
(b) *if the function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ is of moderate growth and the expansion (2.1) contains only terms with positive definite matrices $h \in A_m$, then $\mathcal{H}ol\, F(z) \in \mathcal{M}_m^k(N,\psi)$.*

In both cases for all $g \in \mathcal{S}_m^k(N,\psi)$ the following equality holds:

$$\langle g,\, F\rangle_N = \langle g,\, \mathcal{H}ol\, F\rangle_N \tag{2.6}$$

(see [Pa4], theorem 4.2).

Remark: The cusp form $\mathcal{H}ol\, F$ is uniquely determined by (2.6) under the assumptions of (a), but in (b) this equality is not sufficient to identify the modular form $\mathcal{H}ol\, F$. For example, (2.6) does not change if we replace this modular form by adding to it an Eisenstein series (of Siegel or of Klingen type). Part (a) of Theorem 2.2 was established by Sturm [St].

We now describe a special complex analytic family of the type $\mathcal{H}ol\,(g\cdot G)$ which should correspond in the case of Siegel–Eisenstein series $G = G_k$ to a certain p-adic family coming from the Siegel–Eisenstein measure. We intend to describe in detail this p-adic family in another paper.

For the construction we shall use formulas (2.4) in a special situation described in the next theorem. We intend to apply Theorem 2.3 below in another paper in order to construct some explicit p-adic families of Siegel cusp forms.

2.3. THEOREM: *Suppose the C^∞-modular form $F \in \tilde{\mathcal{M}}_k^m(N,\psi)$ has the form of a product of the type $F(z) = g(z)G(z)$, where*

$$g(z) = \sum_{A_m \ni h > 0} B(h) e_m(hz),$$

$$G(z) = \sum_{A_m \ni h \geq 0} C(h) \det(4\pi y)^{-n} R(4\pi hy; n, \beta) e_m(hz),$$

$F(z)$ satisfies one of the two assumptions (a) *or* (b) *of Theorem 2.2 and $R(z;n,\beta)$ is the polynomial defined for any integer $n \geq 0$, $\beta \in \mathbf{C}$ and $z = {}^tz \in \mathrm{M}_n(\mathbf{C})$ by*

$$R(z;n,\beta) = (-1)^{mn} e^{\operatorname{tr}(z)} \det(z)^{n+\beta} \Delta_m^n [e^{-\operatorname{tr}(z)} \det(z)^{-\beta}],$$

where

$$\Delta_m = \det(\partial_{ij}) \quad (\partial_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial z_{ij}, \quad i \leq j)$$

is the Maass differential operator. Then the following equality holds:

$$\mathcal{H}ol\, F(z) = \sum_{A_m \ni h = h_1 + h_2 > 0} B(h_1) C(h_2) P(h_2, h; n, \beta) e_m(hz), \tag{2.7}$$

where $P(v,u) = P(v,u;n\beta)$ denotes a polynomial of $u = {}^tu = (u_{ij})$ and $v = {}^tv = (v_{ij})$ with the property

$$P(v,u;n,\beta) \equiv \det\, v^n (\operatorname{mod}\langle u_{ij}\rangle)$$

and $P(v,u;n,\beta) \in \mathbf{Q}[u,v]$ for any $\beta \in \mathbf{Q}$.

The proof of Theorem 2.3 is carried out by a straightforward application of the integral formula (2.4) for the action of $\mathcal{H}ol$ on each of the Fourier coefficients of the function $F(z)$:

$$A(y,h) = \sum_{A_m \ni h = h_1 + h_2 > 0} B(h_1) C(h_2) \det(4\pi y)^{-n} R(4\pi h_2 y; n, \beta) e_m(ihz).$$

As a result we get

$$A(h) = \sum_{A_m \ni h = h_1 + h_2 > 0} B(h_1) C(h_2) P(h_2, h; n, \beta),$$

where

$$\begin{aligned} P(v,u) &= P(v,u;n,\beta) \\ &= \frac{\det(4\pi u)^{k-(m+1)/2}}{\Gamma_m(k-(m+1)/2)} \int_Y R(4\pi vy; n,\beta) \det(4\pi y)^{-n} \det\, y^{k-(m+1)/2} e_m(2iuy) d^\times y \\ &= \frac{\det(4\pi u)^{k-(m+1)/2}}{\Gamma_m(k-(m+1)/2)} \int_Y R(4\pi vy; n,\beta) \det(4\pi y)^{-n+k-(m+1)/2} e_m(2iuy) d^\times y \\ &= \frac{\det(4\pi u)^{k-(m+1)/2}}{\Gamma_m(k-(m+1)/2)} \int_Y R(vy; n,\beta) \det(y)^{-n+k-(m+1)/2} e^{-\operatorname{tr} uy} d^\times y \\ &= \frac{\Gamma_m(k-n-(m+1)/2)}{\Gamma_m(k-(m+1)/2)} \det\, u^{k-(m+1)/2} R(v \cdot \partial/\partial u; n,\beta) [\det\, u^{(m+1)/2-k+n}], \end{aligned}$$

with $n \in \mathbf{Z}$, $n \geq 0$, $\beta \in \mathbf{C}$. One can show by differentiation that the function $P(v,u) = P(v,u;n,\beta)$ is a polynomial in u, v with the desired properties.

3. Distributions, measures and non-Archimedean integration

3.1. DISTRIBUTIONS. Let us consider a commutative associative ring R, an R-module $\mathcal{A}$ and a profinite (i.e. compact and totally disconnected) topological space Y. Then Y is a projective limit of finite sets:

$$Y = \varprojlim_{i \in I} Y_i \quad (\pi_{ij} : Y_i \to Y_j, \quad i,j \in I, \ i \geq j) \tag{3.1}$$

where I is a (partially ordered) inductive set and for $i \geq j$, $i,j \in I$ there are surjective homomorphisms $\pi_{i,j}$: $Y_i \to Y_j$ with the condition $\pi_{i,j} \circ \pi_{j,k} = \pi_{i,k}$ for $i \geq j \geq k$. Let $\mathrm{Step}(Y,R)$ be the R-module consisting of all R-valued locally constant functions ϕ: $Y \to R$.

Definition: A distribution on Y with values in a R-module $\mathcal{A}$ is a R-linear homomorphism

$$\mu\colon \mathrm{Step}(Y,R) \to \mathcal{A}. \tag{3.2}$$

For $\varphi \in \mathrm{Step}(Y,R)$ we use the notation

$$\mu(\varphi) = \int_Y \varphi d\mu = \int_Y \varphi(y) d\mu(y).$$

Each distribution μ can be defined by a system of functions $\mu^{(i)}$: $Y_i \to \mathcal{A}$, satisfying the following finite-additivity condition:

$$\mu^{(j)}(y) = \sum_{x \in \pi_{i,j}^{-1}(y)} \mu^{(i)}(x) \quad (y \in Y_j, \ x \in Y_i). \tag{3.3}$$

A criterion that a system of functions $\mu^{(i)}$: $Y_i \to \mathcal{A}$ satisfies the finite-additivity condition (3.3) (and hence is associated to some distribution) is given by the following condition (*compatibility criterion*): for all $j \in I$, and φ_j: $Y_j \to R$ the value of the sums

$$\mu(\varphi) = \mu^{(i)}(\varphi_i) = \sum_{y_i \in Y_i} \varphi_i(y)\mu^{(i)}(y) \quad \text{is independent of } i \tag{3.4}$$

$$\text{for all large enough } i \geq j.$$

Example: The Bernoulli distributions (see [La]). Let M be a positive integer, $f\colon \mathbf{Z} \to \mathbf{C}$ is a periodic function with period M (i.e. $f(x+M) = f(x)$, $f\colon \mathbf{Z}/M\mathbf{Z} \to \mathbf{C}$). The generalized Bernoulli number $B_{k,f}$ is defined by

$$\sum_{k=0}^{\infty} \frac{B_{k,f}}{k!} t^k = \sum_{a=0}^{M-1} \frac{f(a)te^{at}}{e^{Mt}-1}. \tag{3.5}$$

Now let us consider the profinite ring

$$Y = \mathbf{Z}_S = \varprojlim_M \mathbf{Z}/M\mathbf{Z} \quad (S(M) \subset S),$$

the projective limit being taken over the set of all positive integers M with support $S(M)$ in a fixed finite set S of prime numbers. A periodic function $f\colon \mathbf{Z}/M\mathbf{Z} \to \mathbf{C}$ with $S(M) \subset S$ can be regarded as an element of $\mathrm{Step}(Y, \mathbf{C})$ and there exists a distribution $E_k\colon \mathrm{Step}(Y,\mathbf{C}) \to \mathbf{C}$ such that

$$E_k(f) = B_{k,f} \quad \text{for all } f \in \mathrm{Step}(Y, \mathbf{C}). \tag{3.6}$$

For a function $f\colon \mathbf{Z}/M\mathbf{Z} \to \mathbf{C}$ as above let

$$L(s,f) = \sum_{n=1}^{\infty} f(n)n^{-s};$$

then

$$L(1-k,f) = -\frac{B_{k,f}}{k}. \tag{3.7}$$

This implies independence of $B_{k,f}$ on the choice of M. We note also that if $K \subset \mathbf{C}$ is an arbitrary subfield, and $f(Y) \subset K$, then $B_{k,f} \in K$; hence the distribution E_k is a K-valued distribution on Y.

3.2. Measures. Let R be a topological ring, and $\mathcal{C}(Y,R)$ be the topological module of all R-valued functions on a profinite set Y.

Definition: A measure on Y with values in the topological R-module $\mathcal{A}$ is a continuous homomorphism of R-modules

$$\mu\colon \mathcal{C}(Y,R) \to \mathcal{A}.$$

The restriction of μ to the R-submodule $\mathrm{Step}(Y,R) \subset \mathcal{C}(Y,R)$ defines a distribution which we denote by the same letter μ, and the measure μ is uniquely determined by the corresponding distribution since the R-submodule $\mathrm{Step}(Y, R)$ is dense in $\mathcal{C}(Y, R)$.

Now we consider any closed subring R of the Tate field $\mathbf{C}_p$, $R \subset \mathbf{C}_p$, and let $\mathcal{A}$ be a complete R-module with topology given by a norm $|\cdot|_{\mathcal{A}}$ on $\mathcal{A}$ compatible with the norm $|\cdot|_p$ on $\mathbf{C}_p$. Then the fact that a distribution (a system of functions $\mu^{(i)}$: $Y_i \to \mathcal{A}$) gives rise to a $\mathcal{A}$-valued measure on Y is equivalent to the condition that the system $\mu^{(i)}$ is bounded, i.e. for some constant $B > 0$ and for all $i \in I$, $\quad x \in Y_i$ the following uniform estimate holds:

$$|\mu^{(i)}(x)|_{\mathcal{A}} < B. \tag{3.8}$$

3.3. Proposition (The abstract Kummer congruences (see [Ka3], p. 258)): *Let $\{f_i\}$ be a system of continuous functions $f_i \in \mathcal{C}(Y, \mathcal{O}_p)$ in the ring $\mathcal{C}(Y, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group Y with values in the ring of integers $\mathcal{O}_p$ of $\mathbf{C}_p$ such that the $\mathbf{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y, \mathbf{C}_p)$. Let also $\{a_i\}$ be any system of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure μ on Y with the property*

$$\int_Y f_i \, d\mu = a_i$$

is equivalent to the following congruences: for an arbitrary choice of elements $b_i \in \mathbf{C}_p$ almost all of which vanish,

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \quad \text{for all } y \in Y \quad \text{implies} \sum_i b_i a_i \in p^n \mathcal{O}_p. \tag{3.9}$$

3.4. The S-adic Mazur measure. Let $c > 1$ be a positive integer coprime to

$$M_o = \prod_{q \in S} q$$

with S being a fixed set of prime numbers. Using the criterion 3.3 we show that the $\mathbf{Q}$-valued distribution defined by the formula

$$E_k^c(f) = E_k(f) - c^k E_k(f_c), \quad f_c(x) = f(cx), \tag{3.10}$$

turns out to be a measure where $E_k(f)$ are defined by (3.6), $f \in \mathrm{Step}(Y, \mathbf{Q})$ and the field $\mathbf{Q}$ is regarded as a subfield of $\mathbf{C}_p$.

Let us consider next for a complex number $s \in \mathbf{C}$ the distribution μ_s^c on $G_S = \mathbf{Z}_S^\times$ which is uniquely determined by the following condition:

$$\mu_s^c(\chi) = (1 - \chi^{-1}(c) c^{s-1}) L_{M_0}(s, \chi), \tag{3.11}$$

where $L_{M_0}(s,\chi)$ denotes the Dirichlet L-function with Euler factors at primes dividing M_0 removed from its Euler product. Moreover,

$$\mu_{1-k}^{c} = -\frac{E_k^c}{k}$$

is a $\mathcal{O}_p$-measure, and $\mu_0^{(c)}(\chi x_p^r) = \mu_r^c(\chi)$. The corresponding measure $\mu^{(c)} = \mu_0^{(c)}$ is the S**-adic Mazur measure.**

3.5. Non-Archimedean integration. Let S be a finite set of primes containing p. Consider the following $\mathbf{C}_p$-analytic Lie group

$$\mathcal{X}_S = \mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_S, \mathbf{C}_p^\times),$$

where $\mathrm{Gal}_S \xrightarrow{\sim} \mathbf{Z}_S^\times = \prod_{q\in S}\mathbf{Z}_q^\times$ is the Galois group of the maximal abelian extension of $\mathbf{Q}$ unramified outside S and infinity; $\mathbf{C}_p = \overline{\mathbf{Q}}_p$ is the Tate field (completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$). Now we recall the notion of a p-adic measure on Gal_S and properties of its Mellin transform. This Mellin transform is a certain p-adic analytic function on the $\mathbf{C}_p$-analytic Lie group $\mathcal{X}_S$. The canonical $\mathbf{C}_p$-analytic structure on $\mathcal{X}_S$ is obtained by shifts from the obvious $\mathbf{C}_p$-analytic structure on the subgroup

$$\mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times) \subset \mathcal{X}_S.$$

We regard the elements of finite order $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ as Dirichlet characters whose conductor $c(\chi)$ may contain only primes in S, by means of the decomposition,

$$\chi\colon \mathbf{A}_{\mathbf{Q}}^\times/\mathbf{Q}^\times \xrightarrow{\text{class field theory}} \mathrm{Gal}_S \to \overline{\mathbf{Q}}^\times \xrightarrow{i_\infty} \mathbf{C}^\times,$$

where i_∞ is the fixed embedding. The characters $\chi \in \mathcal{X}_S^{\mathrm{tors}}$ form a discrete subgroup $\mathcal{X}_S^{\mathrm{tors}} \subset \mathcal{X}_S$. We shall need also the following natural homomorphism:

$$x_p\colon \mathrm{Gal}_S \to \mathbf{Z}_p^\times \to \mathbf{C}_p^\times, \quad x_p \in \mathcal{X}_S,$$

so that all integers $k \in \mathbf{Z}$ can be regarded as characters of the type $x_p^k\colon y \mapsto y^k$.

Recall that a p-adic measure on Gal_S is a bounded $\mathbf{C}_p$-linear form μ on the space $\mathcal{C}(\mathrm{Gal}_S)$ of all continuous $\mathbf{C}_p$-valued functions

$$\varphi \to \mu(\varphi) = \int_{\mathrm{Gal}_S} \varphi d\mu \in \mathbf{C}_p, \quad \varphi \in \mathcal{C}(\mathrm{Gal}_S),$$

and it is uniquely determined by its restriction to the subspace $\mathcal{C}^1(\mathrm{Gal}_S)$ of locally constant functions. We denote by $\mu(a+(Q))$ the value of μ on the characteristic function of the set

$$a+(Q)=\{x\in \mathrm{Gal}_S \mid x\equiv a \bmod Q\}\subset \mathrm{Gal}_S\,.$$

The Mellin transform L_μ of μ is a bounded analytic function

$$L_\mu\colon \mathcal{X}_S\to \mathbf{C}_p,\quad L_\mu(\chi)=\int_{\mathrm{Gal}_S}\chi d\mu\in \mathbf{C}_p,\quad \chi\in\mathcal{X}_S,$$

on $\mathcal{X}_S$, which is uniquely determined by its values $L_\mu(\chi)$ for the characters $\chi\in \mathcal{X}_S^{\mathrm{tors}}$.

The function

$$L(x)=(1-c^{-1}x(c)^{-1})^{-1}L_{\mu^{(c)}}(x)\quad (x\in\mathcal{X}_S) \tag{3.12}$$

is well-defined and it is holomorphic on $\mathcal{X}_S$ with the exception of a simple pole at the point $x=x_p\in\mathcal{X}_S$. This function is called the *non-Archimedean zeta-function of Kubota–Leopoldt* (see [Ku-Le], [Iw]).

Let $\omega \bmod A$ be a fixed primitive Dirichlet character such that $(A,M_0)=1$ with $M_0=\prod_{q\in S}q$. Put $\overline{S}=S\cup S(A)$, $\overline{M}=\prod_{q\in\overline{S}}q$. Then for any positive integer c with $(c,\overline{M})=1$, $c>1$ there exist $\mathbf{C}_p$-measures $\mu^+(c,\omega),\mu^-(c,\omega)$ on $\mathbf{Z}_S^\times$ which are uniquely determined by the following conditions: for $s\in\mathbf{Z}$, $s>0$

$$i_p^{-1}\left(\int_{\mathbf{Z}_S^\times}\chi x_p^s\,d\mu^+(c,\omega)\right)= \tag{3.13}$$
$$(1-\overline{\chi}\omega(c)c^{-s})\frac{C^s_{\omega\overline{\chi}}}{G(\omega\overline{\chi})}\prod_{q\in S\setminus S(\chi)}\{(1-\chi\overline{\omega}(q)q^{s-1})/(1-\overline{\chi}\omega(q)q^{-s})\}L^+_{M_0}(s,\overline{\chi}\omega),$$

where $S(\chi)$ denotes the support of the conductor of χ, and for $s\in\mathbf{Z}, s\le 0$

$$i_p^{-1}\left(\int_{\mathbf{Z}_S^\times}\chi x_p^s\,d\mu^-(c,\omega)\right)=(1-\chi\overline{\omega}(c)c^{s-1})L^-_{M_0}(s,\overline{\chi},\omega), \tag{3.14}$$

where

$$L^+_{M_0}(s,\overline{\chi}\omega)=L_{\overline{M}}(s,\overline{\chi}\omega)\frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}, \tag{3.15}$$

$$L^-_{M_0}(s,\overline{\chi}\omega)=L_{\overline{M}}(s,\overline{\chi}\omega) \tag{3.16}$$

are the normalized Dirichlet L-functions with $\delta = 0, 1$, $(-1)^\delta = \overline{\chi}\omega(-1)$. The functions (3.13) and (3.14) satisfy the following functional equation:

$$L_{M_0}^-(1-s, \chi\overline{\omega}) = \frac{C_{\omega\overline{\chi}}^s}{G(\omega\overline{\chi})} \prod_{q \in S \setminus S(\chi)} \{(1 - \chi\overline{\omega}(q)q^{s-1})/(1 - \overline{\chi}\omega(q)q^{-s})\} L_{M_0}^+(s, \overline{\chi}\omega),$$

which is equivalent to the standard functional equation of the Dirichlet L-functions with a primitive character.

By definition of μ^c

$$\int_{\mathbf{Z}_S^\times} x\, d\mu^-(c,\omega) = \int_{\mathbf{Z}_{\overline{S}}^\times} x^{-1}\omega\, d\mu^c, \quad \int_{\mathbf{Z}_S^\times} x d\mu^+(c,\omega) = \int_{\mathbf{Z}_{\overline{S}}^\times} x\, x_p^{-1}\omega^{-1}\, d\mu^c,$$

where $x \in \mathcal{X}_S$, and $\mathcal{X}_S$ is regarded as a subgroup of $\mathcal{X}_{\overline{S}}$.

4. p-adic measures defined by the Fourier coefficients of the Siegel–Eisenstein series

4.1. A p-ADIC CONSTRUCTION. Now let S be a set of prime numbers containing a fixed prime number p, $A_{m,S} = A_m \otimes \mathbf{Z}_S$ be a free $\mathbf{Z}_S$-module of rank $m(m+1)/2$ of half integral symmetric matrices of size m over $\mathbf{Z}_S$. We construct certain p-adic measures on the compact group $Y = A_{m,S} \times \mathbf{Z}_S^\times$, with values in the completed semi-group ring $R = \mathcal{O}_p[[q^{B_m}]] \otimes \mathbf{Q}$ of the multiplicative semi-group q^{B_m}, where B_m is the additive group of positive semi-definite half integral matrices $\xi = (\xi_{ij}) \geq 0$, $2\xi_{ij} \in \mathbf{Z}$, $\xi_{ii} \in \mathbf{Z}$, $\mathcal{O}_p$ being the ring of integers of the Tate field $\mathbf{C}_p$. This measure will be characterized by its integrals on the discrete subset $Z_S \subset \mathcal{C}(Y, \mathbf{C}_p)$ formed by elements of the type $((k,\chi),\psi)$ for sufficiently large $k \in \mathbf{Z}$, Dirichlet characters χ: $\mathbf{Z}_S^\times \to \mathbf{C}_p^\times$, and additive characters ψ of $A_{m,S}$.

We use the following simple observation: if for a fixed element

$$G(q) = \sum_{\xi \in B_m} g_\xi q^\xi \in \mathcal{O}_p[[q^{B_m}]] \otimes \mathbf{Q}$$

and for an open compact subset $U \subset A_{m,S}$ we put

$$G(q;U) = \sum_{\xi \in B_m \cap U} g_\xi q^\xi \in \mathcal{O}_p[[q^{B_m}]] \otimes \mathbf{Q}, \tag{4.1}$$

then we obtain a measure μ_G on $A_{m,S}$ defined by $\mu_G(U) = G(q;U)$.

4.2. COEFFICIENTS OF THE NORMALISED EISENSTEIN SERIES AS S-ADIC INTEGRALS. Consider again the normalized Eisenstein series $G^+(z,s) = G^+(z,s;k,\chi,M)$, of Theorem 1.6. Recall that:

For $2s$ to be an integer, $s \le 0, k+2s \ge 1+(m/2)$, $M>1$, we have

$$G^+(z,s;k,\chi,M) = \sum_{A_m \ni h>0} b^+(h,y,s)e_m(hz),$$

where for $s > (m+2-2k)/4$ non-zero terms only occur for positive definite $h>0$, and for all s as above and for $h>0, h \in A_m$ the following identity holds:
(4.2)

$$b^+(h,y,s) = \begin{cases} W^*(y,h,s)L_M^+(k+2s-(m/2),\chi\omega)M(h,\chi,k+2s) & \text{for } m \text{ even}, \\ W^*(y,h,s)M(h,\chi,k+2s) & \text{for } m \text{ odd}, \end{cases}$$

where $\omega = \theta^{m/2}\chi_h$,

$$L^+(s,\chi) = \frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}L_M(s,\chi)$$

is the normalized Dirichlet L-function, $\delta = 0$ or 1 according to $\chi(-1) = (-1)^\delta$, the factor $M(h,\chi,k+2s)$ defined by (1.44), namely

$$M(h,\chi,k+2s) = \prod_{q\in P(h)} M_q(h,\chi(q)q^{-k-2s}), \tag{4.3}$$

is a finite Euler product, extended over primes q in the set $P(h)$ of prime divisors of M and of all elementary divisors of h, where $M_q(h,t) \in \mathbf{Z}[t]$ is a certain polynomial with integral coefficients; χ_h and θ are defined by (1.43) and (1.43a),

$$W^*(y,h,s) = 2^{-m\kappa}\det h^{k+2s-\kappa}\det(4\pi y)^s R(4\pi hy;-s;\kappa-k-s),$$

provided s is an integer, and the polynomial $R(y;n,\beta)$ is defined by (1.33), and otherwise $b^+(h,y,s) = 0$ (i.e. if $2s \in \mathbf{Z}$ but $s \notin \mathbf{Z}$).

Under the above assumption on k we put $s=0$; then according to formulas of Section 2, $R(4\pi hy;-s;\kappa-k-s) = R(4\pi hy;0;\kappa-k) = 1$ and the series $G^+(z,s) = G^+(z,s;k,\chi,M)$ are holomorphic Siegel modular forms with cyclotomic Fourier coefficients:

$$G^+(z,0;k,\chi,M) = \sum_{A_m\ni h>0} b^+(h;k,\chi)e_m(hz),$$

with

$$b^+(h;k,\chi) = \begin{cases} 2^{-m\kappa}\det(h)^{k-\kappa}L_M^+(k-(m/2),\chi\omega)M(h,\chi,k) & \text{for } m \text{ even},\\ 2^{-m\kappa}\det(h)^{k-\kappa}M(h,\chi,k) & \text{for } m \text{ odd}.\end{cases}$$

For the S-adic construction put

$$G_S^+(z;k,\chi,M) = \sum_{\substack{A_m\ni h>0\\ (\det h, M_0)=1}} b^+(h;k,\chi)e_m(hz). \tag{4.4}$$

On the other hand, we have seen in (3.13) that the corresponding values of $L_M^+(k-(m/2),\chi\theta^{m/2}\chi_h)$ can be represented as certain p-adic (or S-adic) integrals. Put in (3.13) $s=k-(m/2)$, $\chi=\overline{\chi}$, and $\omega=\theta^{m/2}\chi_h$; then we have

(4.5)

$$\begin{aligned}&(1-\chi\omega(c)c^{-k+(m/2)})\frac{C_{\omega\chi}^{k-(m/2)}}{G(\omega\chi)}\\ &\times\prod_{q\in S\setminus S(\chi)}\Big\{(1-\overline{\chi}\omega(q)q^{k-(m/2)-1})/(1-\chi\omega(q)q^{-k+(m/2)})\Big\}L_M^+(k-(m/2),\chi\omega)\\ &=i_p^{-1}\left(\int_{\mathbf{Z}_S^\times}\overline{\chi}x_p^{k-(m/2)}\,d\mu^+(c,\omega)\right)\end{aligned}$$

4.3. Theorem: *Let m be even, and suppose that $2\in S$. Let $c>1$ be a positive integer coprime to $M_0=\prod_{q\in S}q$. Then there exists a measure $\mu_{E-S}^{(c)}$ on $Y=\{(\xi,x)\in A_{m,S}\times\mathbf{Z}_S^\times\}$ with values in $R=\mathcal{O}_p[[q^{B_m}]]\otimes\mathbf{Q}$ which is uniquely defined by the following properties: for all pairs (k,χ) with $k\in\mathbf{Z}$ sufficiently large, $2k>m$, and a Dirichlet character $\chi:\mathbf{Z}_S^\times\to\mathbf{C}_p^\times$, $\chi \bmod M$ with M divisible by M_0, one has:*

(a)

(4.6)

$$\begin{aligned}&\int_Y\det(\xi)^{k-\kappa}x_p^{k-(m/2)}\chi(x)d\mu_{E-S}^{(c)}(\xi,x)=\\ &(1-\overline{\chi}(c)^2c^{-2k+m})\frac{C_{\chi}^{k-(m/2)}}{G(\overline{\chi})}\\ &\times\prod_{q\in S\setminus S(\chi)}\{(1-\chi^2(q)q^{2k-m-1})/(1-\overline{\chi}^2(q)q^{-2k+m})\}G_S^+(z;k,\overline{\chi},M)\in R.\end{aligned}$$

(b) *Consider the natural projection $\pi_S\colon Y\to A_{m,S}$. For a fixed pair (k,χ) as above let $\pi_S^*(\mu_{E-S}^{(c)})(x_p^{k-(m/2)}\chi)(\xi)$ denote the measure on $A_{m,S}$ obtained as the* *direct image of* $\mu^{(c)}_{E-S}$ *defined by integrating along the fibers of* π *the function* $x_p^{k-(m/2)}\chi(x)$ *on* $\mathbf{Z}_S^\times$. *Then the measure*

$$\det(\xi)^{k-\kappa}\pi_S^*(\mu^{(c)}_{E-S})(x_p^{k-(m/2)}\chi)(\xi)$$

on $A_{m,S}$ *coincides with the measure* $\mu_{G_S^{+,c}(k,\overline{\chi})}$ *corresponding by (4.1) to the function*

$$G_S^{+,c}(z;k,\overline{\chi}) = (1-\overline{\chi}(c)^2c^{-2k+m})\frac{C_\chi^{k-(m/2)}}{G(\overline{\chi})}$$
$$\times \prod_{q\in S\setminus S(\chi)}\{(1-\chi^2(q)q^{2k-m-1})/(1-\overline{\chi}(q)^2q^{-2k+m})\}G_S^+(z;k,\overline{\chi},M)\in R.$$

In other words, for all locally constant functions $\varphi=\varphi(\xi)$, $\varphi\in \mathrm{Step}(A_{m,S},\mathbf{Q})$ *one has*

$$\int_Y \varphi(\xi)\det(\xi)^{k-\kappa}x_p^{k-(m/2)}\chi(x)d\mu^{(c)}_{E-S}(\xi,x) = \sum_{A_m\ni h>0}\varphi(h)b_S^{+,c}(h;k,\overline{\chi})e_m(hz), \tag{4.7}$$

where $b_S^{+,c}(h;k,\overline{\chi})$ *are the corresponding Fourier coefficients of the function*

$$G_S^{+,c}(z;k,\overline{\chi}) = \sum_{A_m\ni h>0} b_S^{+,c}(h;k,\overline{\chi})e_m(hz)$$

as above.

Remark: The reason of the condition $2\in S$ is that we have to omit the Fourier coefficients containing a nontrivial 2-factor in the finite Euler product (4.3) as we use the work of Shimura [ShDu] and Feit [Fe] who never computed the 2-factor of the Whittaker integral. However, it was pointed out by the referee that for GSp(4) the computation at the prime 2 was carried out by Kaufhold in 1959 [Kauf] so that the condition $2\in S$ could be removed in this case.

4.4. THEOREM: *Let* m *be odd, and suppose that* $2\in S$. *Then there exists a measure* μ_{E-S} *on* $Y=\{(\xi,x)\in A_{m,S}\times\mathbf{Z}_S^\times\}$ *with values in* $R=\mathcal{O}_p[[q^{B_m}]]\otimes\mathbf{Q}$ *which is uniquely defined by the following properties: for all pairs* (k,χ) *with* $k\in\mathbf{Z}$ *sufficiently large,* $2k>m$, *and a Dirichlet character* $\chi:\mathbf{Z}_S^\times\to\mathbf{C}_p^\times$, $\chi \bmod M$ *with* M *divisible by* M_0, *one has:*

(a)

$$\int_Y \det(\xi)^{k-\kappa}x_p^{k-(m/2)}\chi(x)d\mu_{E-S}(\xi,x) = G_S^+(z;k,\overline{\chi},M)\in R, \tag{4.8}$$

(b) *Consider the natural projection* $\pi_S\colon Y \to A_{m,S}$. *For a fixed pair* (k,χ) *as above let* $\pi_S^*(\mu_{E-S})(x_p^{k-(m/2)}\chi)(\xi)$ *denote the measure on* $A_{m,S}$ *obtained as the direct image of* μ_{E-S} *defined by integrating along the fibers of* π *the function* $x_p^{k-(m/2)}\chi(x)$ *on* $\mathbf{Z}_S^\times$. *Then the measure*

$$\det(\xi)^{k-\kappa}\pi_S^*(\mu_{E-S})(x_p^{k-(m/2)}\chi)(\xi)$$

on $A_{m,S}$ *coinsides with the measure* $\mu_{G_S^+(k,\overline{\chi})}$ *corresponding by (4.1) to the function*

$$G_S^+(z;k,\overline{\chi}) = G_S^+(z;k,\overline{\chi},M) \in R.$$

In other words, for all locally constant functions $\varphi = \varphi(\xi)$, $\varphi \in \mathrm{Step}(A_{m,S},\mathbf{Q})$ *one has*

$$\begin{aligned}\int_Y \varphi(\xi)\det(\xi)^{k-\kappa}x_p^{k-(m/2)}\chi(x)d\mu_{E-S}(\xi,x)\\ = \sum_{A_m\ni h>0}\varphi(h)b_S^+(h;k,\overline{\chi})e_m(hz),\end{aligned} \tag{4.9}$$

where $b_S^+(h;k,\overline{\chi})$ *are the corresponding Fourier coefficients of the function*

$$G_S^+(z;k,\overline{\chi}) = \sum_{A_m\ni h>0} b_S^+(h;k,\overline{\chi})e_m(hz)$$

as above.

Remark: Again, the reason for the condition $2 \in S$ is that we have to omit the Fourier coefficients containing a nontrivial 2-factor in the finite Euler product (4.3) as we use the work of Shimura [ShDu] and Feit [Fe] who never computed the 2-factor of the Whittaker integral. However, for GSp(6) it seems that the computation at the prime 2 has been carried out by Katsurada [Kats] so that the condition $2 \in S$ could probably be removed in this case.

4.5. Proof of Theorems 4.3 and 4.4: We use arguments analogous to those in [Pa4], pp. 115–116. It is easier to prove Theorem 4.4, in which case the measure μ_{E-S} is uniquely defined by (4.9) because the functions of the type

$$\varphi(\xi)\det(\xi)^{k-\kappa}x_p^{k-(m/2)}\chi(x)$$

on Y are dense in $\mathcal{C}(Y,\mathbf{C}_p)$. Notice that the right-hand side of (4.9) contains the finite Euler product $M(h,\overline{\chi},k) = \prod_{q\in P(h)} M_q(h,\overline{\chi}(q)q^{-k})$, and it has the form of a finite linear combination of terms of the type $\overline{\chi}(b)b^{-k} = (\chi(b)b^k)^{-1}$

with $(b,p)=1$ whose coefficients are integers independent of χ and k. So one obviously constructs μ_{E-S} term by term. The equality (4.8) then follows from the equality (4.9).

In order to prove Theorem 4.3 we use the integral representation (4.5), which easily transforms to the following:

(4.10)
$$
\begin{aligned}
&(1-\overline{\chi}^2(c)c^{-2k+m})\frac{C_{\omega\overline{\chi}}^{k-(m/2)}}{G(\omega\overline{\chi})}\\
&\times \prod_{q\in S\setminus S(\chi)}\left\{(1-\chi^2(q)q^{2k-m-2})/(1-\overline{\chi}^2(q)q^{-2k+m/2})\right\}L_M^+(k-(m/2),\overline{\chi}\omega)\\
&=\prod_{q\in S\setminus S(\chi)}\left\{(1+\overline{\chi}\omega(q)q^{k-(m/2)-1})/(1+\chi\omega(q)q^{-k+(m/2)})\right\}\\
&\qquad\times i_p^{-1}\left(\int_{\mathbf{Z}_S^\times}\chi x_p^{k-(m/2)}d\mu^+(c,\omega)\right).
\end{aligned}
$$

In order to establish (4.10) we replaced χ by $\overline{\chi}$ and used the identity

$$(1-\chi\omega(q)x)(1+\chi\omega(q)x)=(1-\chi^2(q)x^2). \tag{4.11}$$

Recall that in (4.10) we use the notation ω for the primitive Dirichlet character such that if $\det h=a^2t$ with a square free integer t, then we have that $\chi_h=\chi_t$ where χ_t is the primitive Dirichlet character associated with the quadratic field $\mathbf{Q}(\sqrt{t})$, $(t,S)=1$.

Now we take into account that for $\omega=\theta^{m/2}\chi_t$

$$
\begin{aligned}
&C_{\omega\overline{\chi}}=C_\omega C_{\overline{\chi}},\\
&G(\omega\overline{\chi})=G(\theta^{m/2}\chi_t\overline{\chi})=\omega(C_{\overline{\chi}})\overline{\chi}(C_\omega)G(\theta^{m/2}\chi_t)G(\overline{\chi}).
\end{aligned}
\tag{4.12}
$$

Note also that

$$\omega(C_{\overline{\chi}})=\left(\frac{(-1)^{m/2}t}{C_{\overline{\chi}}}\right). \tag{4.13}$$

We define $\mu_{E-S}^{(c)}$ by (4.7); for this purpose we substitute (4.10) into the definition of the numbers

(4.14)

$$\begin{aligned}
b_S^{+,c}(h;k,\overline{\chi}) =&(1-\overline{\chi}(c)^2c^{-2k+m})\frac{C_\chi^{k-(m/2)}}{G(\overline{\chi})}\\
&\times\prod_{q\in S\setminus S(\chi)}\{(1-\chi^2(q)q^{2k-m-2})/(1-\overline{\chi}(q)^2q^{-2k+m})\}b^+(h;k,\chi)\\
=&(1-\overline{\chi}(c)^2c^{-2k+m})\frac{C_\chi^{k-(m/2)}}{G(\overline{\chi})}\\
&\times\prod_{q\in S\setminus S(\chi)}\{(1-\chi^2(q)q^{2k-m-2})/(1-\overline{\chi}(q)^2q^{-2k+m})\}\\
&\times 2^{-m\kappa}\det(h)^{k-\kappa}L_M^+(k-(m/2),\overline{\chi}\omega)M(h,\overline{\chi},k),
\end{aligned}$$

which are the Fourier coefficients of the right-hand side of (4.7). Taking into account (4.12) and (4.13) we transform (4.14) to the following:

$$\begin{aligned}
&\frac{\omega(C_{\overline{\chi}})G(\omega)}{C_\omega^{k-(m/2)}\chi(C_\omega)}\prod_{q\in S\setminus S(\chi)}\left\{(1+\chi\omega(q)q^{k-(m/2)-1})/(1+\overline{\chi}\omega(q)q^{-k+(m/2)})\right\}\\
&\qquad\times i_p^{-1}\left(\int_{\mathbf{Z}_S^\times}\chi x_p^{k-(m/2)}d\mu^+(c,\omega)\right).
\end{aligned}\tag{4.15}$$

Now using (4.13) we see that the value of $\omega(C_{\overline{\chi}})$ depends only on $\det h \bmod 4M_0$ and we can finish the proof of Theorem 4.3 by subdividing the right-hand side of (4.7) into a finite number of subseries according to $\det h \bmod 4M_0$ and by constructing the S-adic measure of Theorem 4.3 for each of these series term by term as above. Notice that the finite Euler product $M(h,\overline{\chi},k)$ is a finite linear combination of terms of the type $\overline{\chi}(b)b^{-k}=(\chi(b)b^k)^{-1}$ (with $(b,p)=1$) in which coefficients are algebraic integers independent of χ. Then each of the Fourier coefficients of these series as a function of (k,χ) has the form of the following linear combination:

$$\begin{aligned}
&\sum_i A_i\chi(y_i)y_i^k\int_{\mathbf{Z}_S^\times}\chi\,x_p^k\,x_p^{-m/2}d\mu^\pm(c,\omega)=\\
&\sum_i A_i\int_{\mathbf{Z}_S^\times}(\chi\,x_p^k)(y_iy)\,x_p(y)^{-m/2}d\mu^\pm(c,\omega)(y)\quad(y,y_i\in\mathbf{Z}_S^\times),
\end{aligned}\tag{4.16}$$

with uniformly p-adically bounded algebraic coefficients $A_i\in i_p(\mathbf{Q}^{\mathrm{ab}})$. It remains to notice that $x_p(y)^{-m/2}d\mu^\pm(c,\omega)(y)$ is a bounded p-adic measure on $\mathbf{Z}_S^\times$. If we consider the system of functions $\{f_j\}_{j\in J}$, $f_j=\chi_jx_p^{k_j}$ on $\mathbf{Z}_S^\times$, then integration along this measure shows that the abstract Kummer congruences 3.3 for this system of functions are obviously satisfied by the expressions in (4.16) term by term over i, completing the proof.

5. Application: On the Λ-adic Klingen–Eisenstein series

This section was written in cooperation with Koji Kitagawa (Hokkaido University, Japan).

5.0. The purpose of this section is to construct a p-adic measure coming from the Klingen–Eisenstein series on the symplectic group

$$G = \mathrm{GSp}_{2m} = \left\{\alpha \in \mathrm{GL}_{2m} \mid {}^t\alpha J_m \alpha = \nu(\alpha) J_m, \nu(\alpha) \in \mathrm{GL}_1\right\}$$

over $\mathbf{Q}$ where

$$J_m = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

This measure takes values in a space of p-adic Siegel modular forms and defines a $\mathcal{L}$-adic Siegel modular form where $\mathcal{L}$ denotes the field of fractions of the Iwasawa algebra $\Lambda = \mathbf{Z}_p[[X]]$.

More precisely, let $f \in \mathcal{S}_k^r(\Gamma)$ be a cusp form of degree r (with respect to a congruence subgroup Γ of Γ^r of level C). If $k > m + r + 1$ and $m \geq r$ then the Klingen–Eisenstein series is defined as the following absolutely convergent series:

$$E_k^{m,r}(z, f, \chi) = \sum_{\gamma \in \Delta_{m,r} \cap \Gamma \backslash \Gamma} \chi(\det(d_\gamma)) f(\omega^{(r)}(\gamma z)) j(\gamma, z)^{-k}, \tag{5.1}$$

with $z \in \mathfrak{H}^m$, $\omega(z)^{(r)}$ being the upper left corner of z of size $r \times r$,

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

and $\Delta_{m,r}$ denotes the set of elements in Γ^m having the form

$$\begin{pmatrix} * & * \\ 0_{m-r,m+r} & * \end{pmatrix}$$

[Kl]. This series turns out to be a modular form of weight k and of degree m on a congruence subgroup of the group Γ^m. M. Harris proved in [Har2], [Har3] the validity of Garrett's conjecture: all the Fourier coefficients of the modular form $E_k^{m,r}(z, f, \chi)$ belong to the field $\mathbf{Q}(f, \chi)$ generated by the Fourier coefficients of f and χ (at least for trivial χ). Explicit formulas for Fourier coefficients of the series $E_k^{m,r}(z,f)$ were given by Böcherer [Bö2] in general for $k > m+r+1$ and by Kurokawa and Mizumoto [Kur-Miz], [Miz1], [Miz2] who treated the case $m = 2$, $r = 1$. It turned out that the most significant term in these formulas involves the special values of the standard zeta function of f twisted with a certain quadratic Dirichlet character attached to the matrix index ξ of a Fourier coefficient; as noted above, these functions reduce to the (twisted) symmetric squares of the form f if $m = 2$, $r = 1$. The p-adic construction uses the identity of Böcherer [Bö1] generalised by Shimura [Sh95] which expresses these series as certain integrals involving the Petersson product of $f(w)$ with the pullbacks $E(\mathrm{diag}[z,w];k,\chi,N)$ of the Siegel–Eisenstein series (see [Ga])

$$E(Z,k,\chi,N) = \sum_{\alpha\in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\alpha))\det(c_\alpha Z + d_\alpha)^{-k}, \tag{5.2}$$

where Z is a variable in the Siegel upper half plane of degree $n = m + r$,

$$\mathfrak{H}_n = \left\{ Z \in \underset{n}{\mathrm{M}}(\mathbf{C}) \mid {}^tZ = Z = X + iY,\ Y > 0 \right\},$$

$$\Gamma = \Gamma_0^n(N), \quad \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix},$$

and P denotes the subgroup of $P \subset G_{\infty+}$, consisting of elements α with the condition $c_\alpha = 0$, and k is the weight (the above series converges absolutely for $k > n+1$). The series (5.2) may be regarded as a series of the type (5.1): they coinside with $E_k^{m+r,0}(Z,f,\chi)$ with a constant 1 as f.

Let $D(s,f,\chi)$ be the standard zeta function of $f \in \mathcal{S}_k^r(\Gamma)$ as above (with local factors of degree $2r+1$, see [An-K]) and χ be a Dirichlet character. Then the essential fact for our construction is the following identity:

$$\Lambda(k,\chi)D(2k-r,f,\eta)E_k^{m,r}(z,f,\chi) = \langle f'(w), E_k^{m+r,0}(\mathrm{diag}[z,w])\rangle. \tag{5.3}$$

Here $\Lambda(k,\chi)$ is a product of special values of Dirichlet L-functions and Γ-functions, η is a certain Dirichlet character, $E_k^{m+r,0}$ a series of type (5.2) transformed by a suitable element of G^{m+r}, f' an easy transform of f, $(z,w) \in \mathfrak{H}_m \times \mathfrak{H}_r$ (see [Sh95, (7.4), p. 572]). Our construction is based on the fact that the series (5.2) produces a p-adic measure (the Siegel–Eisenstein measure). In the simplest situation this measure depends on the variables (x,ξ) which belong to $\mathbf{Z}_p^\times \times A_{n,p}$, where $\mathbf{Z}_p^\times$ is the p-adic unit group, and $A_{n,p}$ is a free $\mathbf{Z}_p$-module of rank $n(n+1)/2$ formed by half integral symmetric matrices of size n over $\mathbf{Z}_p$. The Siegel–Eisenstein measure defines a Λ-adic modular form whose special values are (involuted) Siegel–Eisenstein series described in Section 3.

5.1. MODULAR FORMS, INVOLUTION AND q-EXPANSION. We put $G^n = \mathrm{Sp}_{2n}(\mathbf{Q})$ and $G^{m,r} = \mathrm{Sp}_{2m}(\mathbf{Q}) \times \mathrm{Sp}_{2r}(\mathbf{Q})$. We define the congruence subgroup $\Gamma_0^n(p^\alpha)$ in the usual way and put $\Gamma_0^{m,r}(p^\alpha) = \Gamma_0^m(p^\alpha) \times \Gamma_0^r(p^\alpha)$. Let B_n be the semigroup of symmetric, semi-definite, half-integral matrices of size n. We put $B_{m,r} = B_m \times B_r$.

Let G be one of the groups G^n $(n = 1,2,3)$ or $G^{2,1}$. Let $\Gamma_0(p^\alpha)$ be the one of $\Gamma_0^n(p^\alpha)$ $(n = 1,2,3)$ or $\Gamma_0^{2,1}(p^\alpha)$ according to G. Let B be the one of B_n $(n = 1,2,3)$ or $B_{2,1}$ according to G;

$$M_k^{2,1}(\Gamma_0^{2,1}(p^\alpha), \psi) =$$
$$\{f \in M_k^{2,1}(\Gamma_1(p^\alpha)) \,\mathrm{St}\, f|_k(g_2, g_1) = \psi(\det(d_2) \cdot d_1) f \text{ for any } (g_2, g_1) \in \Gamma_0^{2,1}(p^\alpha)\}.$$

5.2. Λ-ADIC MODULAR FORMS. Let $\chi = \omega^a$ with some a $(0 \le a < p-1)$, $\omega\colon \mathbf{Z}_p^\times \to \mu_{p-1}$ the Teichmüller character. Let G be one of the groups G^n $(n = 1,2,3)$ or $G^{2,1}$. Let $\Gamma_0(p^\alpha)$ be the one of the groups $\Gamma_0^n(p^\alpha)$ $(n = 1,2,3)$ or $\Gamma_0^{2,1}(p^\alpha)$ according to G. Let B be the one of B_n $(n = 1,2,3)$ or $B_{2,1}$ according to G. Let $\mathcal{P}$ be a Zariski dense subset of $\mathrm{Spec}\,\Lambda(\bar{\mathbf{Q}}_p)$. We call an element F of $\Lambda[[q^B]]$ a Λ-adic modular form on G with character χ with respect to P if $F(\epsilon(u)u^k - 1)$ gives a q-expansion of modular forms in $M_k(\Gamma_0(p^\alpha), \epsilon\chi\omega^{-k})$ for all (k, ϵ) such that $P_{k,\epsilon} \in \mathcal{P}$. We shall take $\mathcal{P} = \mathcal{P}(5) = \{P_{k,\epsilon}; k \ge 5\}$. We denote by $M(G, \chi; \Lambda)$ the Λ-submodule of $\Lambda[[q^B]]$ generated by Λ-adic modular forms on G.

We use the symbol $M^n(\chi; \Lambda)$ $(n = 1,2,3)$ or $M^{2,1}(\chi; \Lambda)$ for $M(G, \chi; \Lambda)$ according to the group G. Then obviously

$$A[[q^{B_{2,1}}]] = A[[q^{B_2}]] \hat{\otimes}_A A[[q^{B_1}]].$$

We let the Hecke operator $T_1(p)$ act on $M^{2,1}(\Gamma_0^{2,1}(p^\alpha), \psi; \mathcal{O})$. We put $e_1 = \lim_{n\to\infty} T_1(p)^{n!}$. We put

$$M^{2,1-ord}(\Gamma_0^{2,1}(p^\alpha), \psi; \mathcal{O}) = e_1 M^{2,1}(\Gamma_0^{2,1}(p^\alpha), \psi; \mathcal{O}).$$

We call $f \in M^{2,1}(\Lambda, \chi)$ 1-ordinary if its specialization at $P_{k,\epsilon}$ is in $M^{2,1-ord}(\Gamma_0^{2,1}(p^\alpha), \psi; \mathcal{O})$ for any $P_{k,\epsilon} \in \mathcal{P}$. We denote by $M^{2,1-ord}(\Lambda, \chi)$ the space of 1-ordinary Λ-adic modular forms, and $M^{ord}(\Lambda, \chi)$ the space of ordinary Λ-adic modular forms in the sense of Hida.

PROPOSITION: $M^{2,1-ord}(\chi; \Lambda) = M^2(\chi; \Lambda) \otimes_\Lambda M^{ord}(\chi; \Lambda)$

Proof: Let $f^i \in M^{ord}(\chi; \Lambda)$ $(i = 1, 2, \ldots, n)$ be a basis. Define Λ-adic linear forms

$$l^i\colon M^{ord}(\chi; \Lambda) \to \Lambda \quad (i = 1, 2, \ldots, n) \quad \text{such that } l^i(f^j) = \delta_{i,j}.$$

Let $P = P_{k,\varepsilon} \in Pc$, $G \in M^{2,1-ord}(\chi; \Lambda)$, $G_P \in M_k^{2,1-ord}(\Gamma_0(p^\alpha), \chi\varepsilon\omega^{-k}; \mathcal{O})$. We have that

$$M_k^{2,1-ord}(\Gamma_0(p^\alpha), \chi\varepsilon\omega^{-k}; \mathcal{O}) =$$
$$M_k^{ord}(\Gamma_0(p^\alpha), \chi\varepsilon\omega^{-k}; \mathcal{O}) \otimes_{\mathcal{O}} M_k^2(\Gamma_0(p^\alpha), \chi\varepsilon\omega^{-k}; \mathcal{O}),$$

and there exist g_P^i such that $G_P = \sum_{i=1}^n f_P^i \otimes g_P^i$. Put

$$R_P^i = l_P^i \otimes id\colon G_P \mapsto (l_P^i \otimes id)\Big(\sum_{j=1}^n f_P^j \otimes g_P^j\Big) = 1 \otimes g_P^i = g_P^i.$$

This gives an $\mathcal{O}$-module homomorphism

$$R_P^i = l_P^i \otimes id\colon M_k^{2,1-ord}(\Gamma_0(p^\alpha), \chi\varepsilon\omega^{-k}; \mathcal{O}) \to M_k^2(\Gamma_0(p^\alpha), \chi\varepsilon\omega^{-k}; \mathcal{O}).$$

We want now to construct a Λ-adic lift of R_P^i or of g_P^i. We apply the following patching lemma to q-expansions of Λ-adic forms, coefficient by coefficient:

LEMMA: *Suppose that for each $P \in \mathcal{P}$ we are given a $g_P \in \Lambda/P\Lambda$ such that they all are compatible (i.e. g_P and g_Q map to the same $g_{P+Q} \in \Lambda/(P+Q)\Lambda$). Then there exist $g \in \Lambda$ such that*

$$\forall P \in \mathcal{P} \quad g_P \equiv g \bmod P \in \Lambda/P\Lambda$$

(see [Wi], [Hi4], p. 232).

Remark: In the definition of Λ-adic forms in 5.2 we assume that the level is just a p-power. This assumption is needed in order to guarantee the semi-simplicity of the Hecke algebra $h^{ord}(\chi, \mathcal{L})$. However, it was pointed out by the referee that the semi-simplicity for a square-free-level Hecke algebra seems to be known now by the work of Coleman–Edixhoven [Co–Ed] so that our construction could also be extended to the square-free-level case. This observation is important also if we wish to keep the condition "$2 \in S$" in Theorem 4.4 (except probably that the Euler 2-factor should then be removed from the L-value).

5.3. THE SIEGEL–EISENSTEIN Λ-ADIC MODULAR FORM. The construction of this form uses normalized Siegel–Eisenstein series. We recall first the definition of the normalized Siegel–Eisenstein series in order to give a precise statement on algebraic properties of these Fourier coefficients. Let χ be a Dirichlet character

$\mathrm{mod} N > 1$ such that $\chi(-1) = (-1)^k$;

$$\begin{aligned} G_k^*(z) =& G^*(z, k, \chi, N) \\ =& N^{nk/2} i^{nk} 2^{-n(k+1)} \pi^{-nk} \Gamma(1_n, 0)^{-1} \\ & \times L_N^*(k, \chi) \prod_{i=1}^{[n/2]} L_N^*(2k - 2i, \chi^2) E(-(Nz)^{-1}) \det(\sqrt{N} z)^{-k} \\ =& N^{nk/2} \tilde{\Gamma}(k, 0) L_N(k, \chi) \prod_{i=1}^{[n/2]} L_N(2k - 2i, \chi^2) E^*(Nz), \end{aligned} \tag{5.4}$$

with

$$\begin{aligned} E^*(Nz) =& E(-(Nz)^{-1}) \det(Nz)^{-k} = N^{-kn/2} E|W(N), \\ \tilde{\Gamma}(k, 0) =& i^{nk} 2^{-n(k+1)} \pi^{-nk} \Gamma(1_n, 0)^{-1} \Gamma((k + \delta)/2) \prod_{j=1}^{[n/2]} \Gamma(k - j) \\ =& i^{nk} 2^{-n(k+1)} \pi^{-nk} \times \begin{cases} \Gamma_n(k) \Gamma(k - \frac{1}{2} n + \mu)/2), & \text{if } n \text{ is even;} \\ \Gamma_n(k), & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{5.5}$$

In this article we are interested only in the case when n is odd. Then one has

$$\begin{aligned} G_k^+(z) =& G_k^-(z) = G_k^*(z) \\ =& i^{nk} 2^{-n(k+1)} \pi^{-nk} \Gamma_n(k) L_N(k, \chi) \prod_{i=1}^{[n/2]} L_N(2k - 2i, \chi^2) E|W(N) \end{aligned} \tag{5.6}$$

where

$$\Gamma_n(s) = \pi^{n(n-1)/4} \prod_{j=0}^{n-1} \Gamma(s - (j/2)). \tag{5.7}$$

Here

$$W(N) = \begin{pmatrix} 0_n & -1_n \\ N1_n & 0_n \end{pmatrix}$$

denotes the principal involution of level N, and the gamma factor $\Gamma(1_n, s)$ is defined in Proposition 1.4.

We now pass to a p-adic construction. Let $A_{n,p} = A_n \otimes \mathbf{Z}_p$ be a free $\mathbf{Z}_p$-module of rank $n(n+1)/2$ of half integral symmetric matrices of size n over $\mathbf{Z}_p$. We construct certain p-adic measures on the compact group $Y = A_{n,p} \times \mathbf{Z}_p^\times$, with values in the completed semi-group ring $R = \mathcal{O}_p[[q^{B_n}]] \otimes \mathbf{Q}$ of the multiplicative semi-group q^{B_n}, where B_n is the additive group of positive semi-definite, half-integral matrices $\xi = (\xi_{ij}) \geq 0$, $2\xi_{ij} \in \mathbf{Z}$, $\xi_{ii} \in \mathbf{Z}$, $\mathcal{O}_p$ being the ring of integers of the Tate field $\mathbf{C}_p$. This measure will be characterized by its integrals on the discrete subset $Z_p \subset Y$ formed by elements of the type $((k, \chi), \psi)$ for sufficiently large $k \in \mathbf{Z}$, Dirichlet characters χ on $\mathbf{Z}_p^\times$, and additive characters ψ of $A_{n,p}$. We use the following simple observation: if for a fixed element

$$G(q) = \sum_{\xi \in B_n} g(\xi) q^\xi \in \mathcal{O}_p[[q^{B_n}]] \otimes \mathbf{Q}$$

and for an open compact subset $U \subset A_{n,p}$ we put

$$G(q; U) = \sum_{\xi \in B_n \cap U} g(\xi) q^\xi \in \mathcal{O}_p[[q^{B_n}]] \otimes \mathbf{Q}, \tag{5.8}$$

then we obtain a measure μ_G on $A_{n,p}$ defined by $\mu_G(U) = G(q; U)$. Now we observe that the coefficients of the normalised Eisenstein series can be represented as certain p-adic integrals.

Consider again the normalized Eisentein series $G_k^+(z) = G^+(z, k, \chi, N)$. Under the above assumption on k these series are holomorphic Siegel modular forms with cyclotomic Fourier coefficients:

$$G^+(z, k, \chi, N) = \sum_{A_n \ni \xi > 0} b^+(\xi, k, \chi) e_n(\xi z),$$

with

(5.9)

$$b^+(\xi, k, \chi) = \begin{cases} 2^{-n\kappa} \det(\xi)^{k-\kappa} L_M^+(k - (n/2), \chi\omega_\xi) M(\xi, \chi, k) & \text{for } n \text{ even}, \\ 2^{-n\kappa} \det(\xi)^{k-\kappa} M(\xi, \chi, k) & \text{for } n \text{ odd}. \end{cases}$$

Here $\kappa = (n+1)/2$, and $L_M^+(k - (n/2), \chi\omega_\xi)$ is the value of a normalised Dirichlet L-function, and the integral factor

$$M(\xi, \chi, k) = \prod_{q \in P(\xi)} M_q(\xi, \chi(q) q^{-k}) \tag{5.10}$$

is a finite Euler product, extended over primes q in the set $P(\xi)$ of prime divisors of the number N and of all elementary divisors of the matrix ξ. The important property of the product is that for each q we have that $M_q(\xi, t) \in \mathbf{Z}[t]$ is a polynomial with integral coefficients. Its explicit form is not important for our construction.

For the p-adic constuction we put $M = Np$ and

$$G_p^+(z,k,\chi,M) = \sum_{\substack{A_n \ni \xi > 0 \\ (\det \xi, M)=1}} b^+(\xi,k,\chi) e_n(\xi z). \tag{5.11}$$

Recall that according to Theorem 4.4 we have that for n odd there exists a measure μ_{E-S} on $Y = \{(\xi, x) \in A_{n,p} \times \mathbf{Z}_p^\times\}$ with values in $R = \mathcal{O}_p[[q^{B_n}]] \otimes \mathbf{Q}$ which is uniquely defined by the following properties: for all pairs (k,χ) with $k \in \mathbf{Z}$ sufficiently large, $2k > n$, and a Dirichlet character $\chi : \mathbf{Z}_p^\times \to \mathbf{C}_p^\times$, χ mod M with M divisible by p, one has

$$\int_Y \det(\xi)^{k-\kappa} x_p^{k-(n/2)} \chi(x) \mu_{E-S}(\xi, x) = G_p^+(z,k,\chi^{-1},M) \in R. \tag{5.12}$$

Proof of the theorem is given in 4.5 and uses arguments analogous to those in [Pa4], pp. 115–116. Notice that the Fourier coefficients of the right-hand side of (5.12) contain the finite Euler product $M(\xi,\overline{\chi},k) = \prod_{q\in P(\xi)} M_q(\xi, \overline{\chi}(q)q^{-k})$, and it has the form of a finite linear combination of terms of the type $\overline{\chi}(b)b^{-k} = (\chi(b)b^k)^{-1}$ with $(b,p) = 1$ whose coefficients are integers independent of χ and k. So one obviously constructs μ_{E-S} term by term.

The explicit formulas for the Fourier coefficients show that this measure takes values in $\mathbf{Z}_p[[q^{B_n}]]$ and its restriction to $\mathbf{Z}_p^\times \xrightarrow{\sim} \Delta \times \Gamma$, $\Gamma = \langle u \rangle$ defines an element $E_\Lambda(\chi) \in M^3(\chi;\Lambda)$ which is a power series in $\Lambda[[q^{B_n}]]$ and which is called the Siegel–Eisenstein Λ-adic modular form whose special values $X = \varepsilon(u)u^k - 1$ are given in terms of (5.12). More precisely, we normalize this Λ-adic form by the condition

$$E_\Lambda(\chi)(\varepsilon(u)u^k - 1) = G_p^+(z,k,\chi^{-1}\varepsilon^{-1}\omega^k, M) \quad (M = p^\alpha).$$

5.4. The pullback of Λ-adic modular forms. Let $\pi\colon B_3 \to B_2 \times B_1$ be a map given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, a_{33} \right).$$

We note that for each $(\xi_2,\xi_1) \in B_{2,1}$ the fiber $\pi^{-1}(\xi_2,\xi_1)$ is a finite set. For $F = \sum a(\xi_3,F)q^{\xi_3} \in \Lambda[[q^{B_3}]]$ we define, $\Phi(F) \in \Lambda[[q^{B_{2,1}}]]$,

$$a(\xi_2,\xi_1,\Phi(F)) = \sum_{\xi_3 \in \pi^{-1}(\xi_2,\xi_1)} a(\xi_3, F);$$

Φ is a Λ-module homomorphism.

PROPOSITION: *$\Phi^3_{2,1}$ maps $M^3(\chi;\Lambda)$ into $M^{2,1}(\chi;\Lambda)$.*

Let $E_\Lambda(\chi) \in M^3(\chi;\Lambda)$ be the (involuted) Λ-adic Siegel–Eisenstein series of degree 3 with character χ constructed in 4.5. Then $\Phi^3_{2,1}(E_\Lambda(\chi)) \in M^{2,1}(\chi;\Lambda)$.

5.5. Λ-ADIC PETERSON INNER PRODUCT. We follow here Hida's idea of algebraic Peterson inner product ([Hi4, p. 222]). Let $S^{ord}(\chi;\Lambda)$ denote the space of Λ-adic cusp forms with character χ. Let $h^{ord}(\chi;\Lambda)$ be the p-adic Hecke algebra with character χ. We note that $S^{ord}(\chi;\Lambda)$ and $h^{ord}(\chi;\Lambda)$ are the Λ-dual of each other by the pairing

$$\langle f,h\rangle = a(1,f|h).$$

We put

$$D = h^{ord}(\chi;\mathcal{L}) = h^{ord}(\chi;\Lambda)\otimes_\Lambda \mathcal{L}.$$

Since D is semi-simple there is a non-degenerate pairing $(\cdot,\cdot)$ on D given by

$$(h,g) = Tr_{D/\mathcal{L}}(hg).$$

By duality we have the dual paring $(\cdot,\cdot)$ on $S^{ord}(\chi;\mathcal{L})$. If $F \in S^{ord}(\chi;\Lambda)$ is the normalized eigenform, then

$$c(F,G) = \frac{(F,G)}{(F,F)} \in \mathcal{L}$$

is well-defined. We write $M(\chi;\Lambda) = M^1(\chi;\Lambda)$. Let $M^{ord}(\chi;\Lambda)$ denote the ordinary part and let π_{ord} denote the projection to the ordinary part. The Hecke algebra $H^{ord}(\chi;\Lambda)$ acts on $M^{ord}(\chi;\Lambda)$; $H^{ord}(\chi;\mathcal{L}) = H^{ord}(\chi;\Lambda)\otimes_\Lambda\mathcal{L}$ is a direct sum of $h^{ord}(\chi;\mathcal{L})$ and the Hecke algebra over $\mathcal{L}$ corresponding to Eisenstein series. Let 1_{cusp} denote the idempotent corresponding to $h^{ord}(\chi;\mathcal{L})$. We define a Λ-bilinear map $\langle\cdot,\cdot\rangle\colon S^{ord}(\chi;\Lambda)\times M(\chi;\Lambda)\to\mathcal{L}$ by

$$\langle F,G\rangle = (F, 1_{cusp}\cdot\pi_{ord}(G)).$$

5.6. Λ-ADIC KERNEL FUNCTION. Let $G \in M^{2,1}(\chi;\Lambda)$. We write

$$G = \sum_{j\in J} g_j\otimes h_j$$

with $g_j \in M^2(\chi;\Lambda)$ and $h_j \in M^1(\chi;\Lambda)$. We define a Λ-linear map $R(G)$: $S^{ord}(\chi;\Lambda)\to M^2(\chi;\Lambda)\otimes\mathcal{L}$ by

$$R(G) = \sum_{j\in J} g_j\langle\cdot,h_j\rangle. \tag{5.13}$$

In the definition (5.13) we used the Λ-bilinear map $\langle\cdot,\cdot\rangle\colon S^{ord}(\chi;\Lambda)\times M(\chi;\Lambda)\to\mathcal{L}$ of the previous section.

5.7. THE Λ-ADIC BÖCHERER–SHIMURA FORMULA. We may call

$$E_\Lambda(F,\chi) = \frac{1}{\langle F,F\rangle} R(\Phi(E_\Lambda(\chi))(F) \in M^2(\chi;\Lambda)\otimes_\Lambda \mathcal{L}$$

the Λ-adic Klingen–Eisenstein series for a Λ-adic cusp form $F \in S^{ord}(\chi;\Lambda)$.

Now our task is to investigate a precise relation of

$$\frac{R(\Phi(E_\Lambda(\chi))(F)}{(F,F)} \tag{5.14}$$

with the classical Klingen–Eisenstein series. We give here only an expected formula:

Let $m=2$, $r=1$. For all $P=P_{k,\varepsilon}\in\mathcal{P}$ we have that the specialization of (5.14) at P is given by

$$\begin{aligned}&\Lambda_p(k,\omega^{-k}\chi\varepsilon)\frac{D(2k-r,f_{k,\varepsilon})}{\langle f_{k,\varepsilon},f_{k,\varepsilon}\rangle}E_k^{m,r}(z,f_{k,\varepsilon},\chi) =\\ &\Lambda_p(k,\omega^{-k}\chi\varepsilon)\frac{\langle f'_{k,\varepsilon}(w),E_k^{m+r,0}(\mathrm{diag}[z,w],\omega^{-k}\chi\varepsilon)\rangle}{\langle f_{k,\varepsilon},f_{k,\varepsilon}\rangle},\end{aligned} \tag{5.15}$$

where $\Lambda_p(k,\omega^{-k}\chi\varepsilon)$ is an explicitly given elementary factor.

Proof of (5.15) (in case $m=2$, $r=1$) is based on the identity (5.3), the definition of $E_\Lambda(\chi)$, of the kernel function

$$R(G)\colon S^{ord}(\chi;\Lambda)\to M^2(\chi;\Lambda)\otimes\mathcal{L}$$

of 5.6.

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